JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 31, No. 4, July–August 2008

Optimal Impulsive Relative Orbit Transfer Along a Circular Orbit

Yoshihiro Ichimura* and Akira Ichikawa†
Kyoto University, Kyoto 606-8501, Japan

DOI: 10.2514/1.32820

The relative orbit transfer problem associated with the Hill–Clohessy–Wiltshire equations is considered, and the open-time minimum-fuel problem with impulsive control is formulated. In particular, between two elliptic relative orbits of the in-plane motion, optimal three-impulse controllers are constructed. For the out-of-plane motion, optimal single-impulse strategies are obtained. Based on these results and the null controllability with the vanishing energy of the Hill–Clohessy–Wiltshire equations, a design method of feedback controllers with a total velocity change close to the optimal one is proposed. It is shown by simulation results that asymptotic relative orbit transfer is fulfilled by the proposed feedback controllers.

Nomenclature

A_i	=	state matrix
a	=	size parameter of the in-plane orbit
B_i	=	control matrix
b	=	size parameter of the out-of-plane orbit
c	=	drift parameter
d	=	deviation parameter
G	=	universal gravitational constant
h_c	=	height of circular orbit
$J(\mathbf{u}, \mathbf{x}_0)$	=	quadratic cost
M_e	=	mass of the Earth
N	=	set of natural numbers
n	=	orbit rate
Q	=	penalty matrix on state
R	=	penalty matrix on control
R_0	=	distance between the target spacecraft and Earth
T	=	period of orbit
T_s	=	settling time
$\mathbf{u}$	=	input vector
ΔV_T	=	total velocity change
ΔV_T^*	=	minimum total velocity change
$\mathbf{x}$	=	state vector
(x, y, z)	=	coordinate frame fixed in target spacecraft
Z_+	=	set of nonnegative integers
α	=	initial phase on the in-plane orbit
β	=	initial phase on the out-of-plane orbit
μ_e	=	gravitational parameter of Earth

I. Introduction

THE relative motion of one spacecraft (chaser) with respect to another (target) in a circular orbit is described by autonomous nonlinear differential equations. The linearized equations are referred to as the Hill–Clohessy–Wiltshire (HCW) equations [1–3]. They are useful for spacecraft rendezvous and docking [4–8], as well as for spacecraft formations [9–14]. The solutions of the HCW equations of the in-plane motion lie on either fixed ellipses or drifting ellipses, which are characterized by three parameters representing the size, the deviation of the center from the target, and the drift. When the drift parameter is zero, the solutions become periodic. They form elliptic relative orbits of the chaser and are exploited for passive rendezvous and formation flight, because no energy is required to keep the chaser in these orbits. Such orbits could be used as temporary orbits before a mission. Elliptic orbits of small size would be convenient for proximity operations such as inspection and repair. For long-term space missions, a series of operations is generally planned, and relative orbit transfers are necessary when the operation of a spacecraft changes. Therefore, it is useful to consider the relative orbit transfer problem and to develop a good control strategy for the transfer.

The Tschauner–Hempel equations replace the HCW equations if the target lies in an eccentric orbit, and they are also used for rendezvous and formation flight [7,15–18].

For rendezvous problems, impulsive maneuvers are assumed and minimum-fuel trajectories are investigated, for which the fixed time and/or fixed end conditions are usually assumed [4–7,16,17,19]. The optimal solutions are obtained by solving two-point boundary problems. In the case of a circular reference orbit, Prussing [4,5] considered a fixed time problem and derived two-, three-, and four-impulse solutions, whereas Jezewski and Donaldson [6] considered the two-impulse problem with free time and fixed end conditions. Carter [7] and Carter and Brient [16,17] assumed fixed time and fixed end conditions and derived the necessary and sufficient conditions for the optimal solution for both circular and elliptic orbits. For rendezvous and transfer problems, the number of impulses needed is known, which is at most equal to the dimension of the state space [17,20–22]. Thus, for an optimal planar rendezvous based on linearized equations of motion, at most four impulses are necessary. Impulsive maneuvers are also used for formation flight [14,23] and, in particular, a linear quadratic regulator (LQR) [10–12] is employed for formation keeping [9,24]. Impulsive maneuvers are also employed for the initialization problem, for which the initial conditions for the nonlinear equations of relative motion are adjusted to those corresponding to periodic solutions, thus making the relative motion bounded [23,25]. It is shown that a single impulse is sufficient for the initialization of the relative motion between Keplerian elliptic orbits, and the impulse with minimum velocity change is obtained [25].

Recently, it has been shown that the HCW equations and the Tschauner–Hempel equations have the property of null controllability with vanishing energy (NCVE) [26]. According to this property, any state of the system can be steered to the origin with an arbitrarily small amount of control energy in the L^2 (square integral) sense [27]. Using this property, the relative orbit transfer problems by continuous-time feedback controllers with small L^2 norms were studied by Shibata and Ichikawa [26]. Feedback controllers are

Received 15 June 2007; revision received 3 December 2007; accepted for publication 28 January 2008.

*Graduate Student, Department of Aeronautics and Astronautics; currently Engineer, Nagoya Guidance and Propulsion Systems Works, Mitsubishi Heavy Industries, Ltd., Komaki 485-8561, Japan.

†Professor, Department of Aeronautics and Astronautics; ichikawa@kuaero.kyoto-u.ac.jp.

designed through algebraic and differential Riccati equations with a small penalty matrix on state.

In this paper, the relative orbit transfer problem associated with the HCW equations is considered, for which the impulsive controls are assumed and the minimum-fuel problem is formulated. Because the in-plane and out-of-plane motion are decoupled, they are considered separately. The trajectories of the in-plane motion are characterized by three parameters, (a, d, c), where a is the size of the ellipse, d is the deviation of the center of the ellipse from the origin, and c is the magnitude of the drift velocity. If $c \neq 0$, (a, d, c) is a drifting ellipse. If $c = 0$, $(a, d, 0)$ is a stationary ellipse, and the solution to it is periodic. The relative orbit transfer from (a_0, d_0, c_0) to (a_f, d_f, c_f) is described as follows. The chaser is initially at a given point of the orbit (a_0, d_0, c_0). A finite number of impulse times and impulsive velocity changes are then sought, which steer the chaser to the final orbit (a_f, d_f, c_f) such that after the final impulse the chaser maintains the orbit. The minimum-fuel problem is to find a finite-impulse strategy that minimizes the total velocity change required for the transfer. In this problem, the initial condition is fixed, the final condition is restricted to the final orbit but is otherwise free, the number and the instants of the impulses are arbitrary and, hence, the final time is free. This is an open-time problem and, to the best of authors' knowledge, it has not been discussed in the literature. For this problem, the usual optimal control and optimization theories are not applicable and so, for example, the primer vector theory is not applicable. The parameterization of elliptic relative orbits is also used by Campbell [13], and the minimum-time and minimum-fuel problems with a fixed final time by continuous controls are considered. As for the out-of-plane motion, the relative orbit is parameterized by a single size parameter, and the relative orbit transfer problem in the phase plane is formulated.

To solve the minimum-fuel problem for the in-plane motion, an elementary but completely different analysis is used. First, two lower bounds for the minimum total velocity change are established by changing the parameter a or c by m impulses. Then, admissible impulse strategies that attain the lower bound are sought. If the difference of the size parameters is large compared with that of the drift parameters, optimal three-impulse strategies can be constructed. It is shown that the impulse times for the optimal strategies are those instants at which the velocity in the radial direction is zero. If the difference of drift parameters is relatively large, only the suboptimal strategies with two or three impulses whose total velocity changes are arbitrarily close to the lower bound are assured. The minimum or infimum of the total velocity change is explicitly given in terms of the size parameter or the drift parameter. The relative orbit transfer problem for the out-of-plane motion is much simpler; an optimal single impulse exists, and its impulse times are those instants that the chaser crosses the orbit plane of the target.

Once the minimum-fuel problem is solved, the next task is to design feedback controllers for the nonlinear equations of the relative motion. Introducing impulse times, which accommodate the optimal impulse times, to the HCW equations, a discrete-time system is derived. Using this system, feedback controllers whose total velocity change to keep the final relative orbit as close to optimal are designed. This is done by making use of NCVE [28,29], which this system inherits from the HCW equations. The NCVE property suggests the use of LQR theory with a small penalty matrix on the state. This generally leads to the desired feedback controllers. A discretized system together with the LQR theory has been used for formation keeping [9] by impulse controls, for which the HCW equations with constant disturbances are considered. The LQR theory is also used for formation flying with pulse-based controllers [11]. The minimum-fuel problem in the L^2 sense for the relative orbit transfer was recently considered by Palmer [30], in which the final time is fixed. But the design of our feedback controllers has two new features: 1) the impulse times are determined from those of the optimal impulse strategies, and 2) the penalty matrix is selected from the NCVE point of view and is, therefore, theoretically supported. Similarly, feedback controllers for out-of-plane motion are designed. The relative orbit transfer problem by feedback in this paper is essentially the same as that of Shibata and Ichikawa [26], in which continuous control is used.

For numerical simulations, a circular orbit of height 500 km is considered, and relative orbit transfer problems are solved. Feedback controllers for both in-plane and out-of-plane motion are designed. They are applied to the nonlinear equations, and the total velocity changes are calculated. Combined controllers are then applied to the three-dimensional nonlinear equations of relative motion. A few other performance indices, such as the maximum values of impulses and settling times, are also calculated. Simulation results indicate that feedback controllers with good performance can be designed by the proposed approach.

This paper is organized as follows. Section I is the Introduction. Section II reviews the HCW equations, provides the characterization of the relative orbits, and gives the formulation of the minimum-fuel problem for relative orbit transfer. In Sec. III, two lower bounds for the minimum total velocity change for in-plane and out-of-plane motion are derived. In Sec. IV, the relative orbit transfer problem by impulsive open-loop control is considered. For in-plane motion, optimal three-impulse strategies are constructed when the difference of the size parameter is larger than that of the drift parameters. Otherwise, suboptimal two- or three-impulse strategies are obtained. Based on these strategies, a discrete-time system is derived in Sec. V, and the design method of impulsive feedback controllers based on NCVE is proposed. In Sec. VI, simulation results are given to illustrate the theory and the effectiveness of the proposed controllers.

II. Equations of Relative Motion and Minimum-Fuel Problem

In this section, the HCW equations, which describe the relative motion of the chaser with respect to the target, will be briefly reviewed.

Consider the target spacecraft in a circular orbit of radius R_0. The orbit rate and the period in this case are given, respectively, by $n = (\mu_e/R_0^3)^{1/2}$ and $T = 2\pi/n$, where $\mu_e \equiv GM_e$ is the gravitational parameter of the Earth, G is the universal gravitational constant, and M_e is the mass of the Earth. To introduce the HCW equations, a coordinate system (x, y, z), fixed at the center of mass of the target, is employed, where the x axis is along the radial direction, the y axis is along the flight direction of the target, and the z axis is out of the orbit plane and completing the right-hand reference frame. Newton's equation of motion then yields the following three equations:

$$\ddot{x} = 2n\dot{y} + n^2(R_0 + x) - \frac{\mu_e(R_0 + x)}{[(R_0 + x)^2 + y^2 + z^2]^{3/2}} + u_x \tag{1}$$

$$\ddot{y} = -2n\dot{x} + n^2 y - \frac{\mu_e y}{[(R_0 + x)^2 + y^2 + z^2]^{3/2}} + u_y \tag{2}$$

$$\ddot{z} = -\frac{\mu_e z}{[(R_0 + x)^2 + y^2 + z^2]^{3/2}} + u_z \tag{3}$$

where (u_x, u_y, u_z) are the thrust accelerations. The linearized equations

$$\ddot{x} = 3n^2 x + 2n\dot{y} + u_x \tag{4}$$

$$\ddot{y} = -2n\dot{x} + u_y \tag{5}$$

$$\ddot{z} = -n^2 z + u_z \tag{6}$$

are well known [3] and are referred to as Hill–Clohessy–Wiltshire equations.

Equations (4) and (5) and Eq. (6) are independent. The former describes the in-plane motion and the latter the out-of-plane motion.

A. In-Plane Motion

The state equation of the in-plane motion is given by

$$\dot{\mathbf{x}} = A_1\mathbf{x} + B_1\mathbf{u} \tag{7}$$

where $\mathbf{x} = [\,x \quad y \quad \dot{x} \quad \dot{y}\,]^T$, $\mathbf{u} = [\,u_x \quad u_y\,]^T$, and

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & -2n & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This system is NCVE because it is controllable and the eigenvalues of A are $(0, 0, in, -in)$ [26]. In fact, recall that a linear system is NCVE if any state can be steered to the origin by a control with an arbitrarily small L^2 norm, and the necessary and sufficient conditions for this are that the system is controllable and all eigenvalues of the system matrix have nonpositive real parts [27]. The state transition matrix $e^{A_1 t}$ is given by [3]

$$e^{A_1 t} = \begin{bmatrix} 4 - 3\cos nt & 0 & (1/n)\sin nt & (2/n)(1 - \cos nt) \\ 6(\sin nt - nt) & 1 & (2/n)(\cos nt - 1) & (4/n)\sin nt - 3t \\ 3n \sin nt & 0 & \cos nt & 2\sin nt \\ 6n(\cos nt - 1) & 0 & -2\sin nt & 4\cos nt - 3 \end{bmatrix} \tag{8}$$

If $\mathbf{u} = 0$, the state for a given initial condition $\mathbf{x}(t_0) = [\,x_0 \quad y_0 \quad \dot{x}_0 \quad \dot{y}_0\,]^T$ is expressed as

$$\begin{aligned} x(t) &= 4x_0 + 2\dot{y}_0/n - (3x_0 + 2\dot{y}_0/n)\cos n(t - t_0) \\ &\quad + (\dot{x}_0/n)\sin n(t - t_0), \\ y(t) &= y_0 - 2\dot{x}_0/n + (2\dot{x}_0/n)\cos n(t - t_0) \\ &\quad + (6x_0 + 4\dot{y}_0/n)\sin n(t - t_0) - (6nx_0 + 3\dot{y}_0)(t - t_0), \\ \dot{x}(t) &= \dot{x}_0 \cos n(t - t_0) + (3nx_0 + 2\dot{y}_0)\sin n(t - t_0), \\ \dot{y}(t) &= (6nx_0 + 4\dot{y}_0)\cos n(t - t_0) - 2\dot{x}_0 \sin n(t - t_0) \\ &\quad - (6nx_0 + 3\dot{y}_0) \end{aligned} \tag{9}$$

It is parameterized by four constants a, d, c, and α as

$$\begin{aligned} x(t) &= 2c + a\cos[n(t - t_0) + \alpha], \\ y(t) &= d - 3nc(t - t_0) - 2a\sin[n(t - t_0) + \alpha], \\ \dot{x}(t) &= -an\sin[n(t - t_0) + \alpha], \\ \dot{y}(t) &= -3nc - 2an\cos[n(t - t_0) + \alpha] \end{aligned} \tag{10}$$

where

$$\begin{aligned} &a = [(3x_0 + 2\dot{y}_0/n)^2 + (\dot{x}_0/n)^2]^{1/2}, \qquad d = y_0 - 2\dot{x}_0/n, \\ &c = 2x_0 + \dot{y}_0/n, \qquad \cos\alpha = -(1/a)(3x_0 + 2\dot{y}_0/n), \\ &\sin\alpha = -\dot{x}_0/(na) \end{aligned} \tag{11}$$

Moreover,

$$\left(\frac{x(t) - 2c}{a}\right)^2 + \left(\frac{y(t) - d + 3nc(t - t_0)}{2a}\right)^2 = 1 \tag{12}$$

When $c = 0$, the trajectory is periodic with period $T = 2\pi/n$ and forms an ellipse with center $(0, d)$ and eccentricity $e = \sqrt{3}/2$. Because the parameters a and d represent the size of the ellipse and the deviation of the center of the ellipse from the origin, respectively, they are referred to as the size parameter and deviation parameter, respectively. The parameter α indicates the initial position on the ellipse as shown in Fig. 1, and it will be called the initial phase of the chaser. If the relative orbits encircle the target spacecraft, they are useful for rendezvous and passive flyaround. If $c \neq 0$, y contains the drift term $-3nc(t - t_0)$. Hence, c is referred to as the drift parameter. In this case (x, y) lies on a drifting ellipse, in which the center of the ellipse moves in the y direction with constant velocity $-3nc$. Thus, d indicates the initial deviation of the center at time t_0.

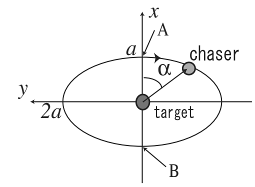


Fig. 1 Elliptic orbit.

If an impulse input $\Delta\dot{\mathbf{x}}_j = [\,\Delta\dot{x}_j \quad \Delta\dot{y}_j\,]^T$ is introduced at $t = t_j$ to Eqs. (4) and (5), it will change the velocity instantaneously and

$$\dot{x}(t_j^+) = \dot{x}(t_j) + \Delta\dot{x}_j, \qquad \dot{y}(t_j^+) = \dot{y}(t_j) + \Delta\dot{y}_j \tag{13}$$

where $\dot{x}(t_j^+) = \lim_{t\downarrow t_j} \dot{x}(t)$. Then Eq. (7) becomes

$$\dot{\mathbf{x}} = A_1\mathbf{x} \qquad (t \neq t_j), \qquad \Delta\mathbf{x}(t_j) = B_1\Delta\dot{\mathbf{x}}_j \tag{14}$$

where $\Delta\mathbf{x}(t_j) = \mathbf{x}(t_j^+) - \mathbf{x}(t_j)$,

$$\mathbf{x}(t_j^+) = \lim_{t\downarrow t_j} \mathbf{x}(t)$$

Suppose that the initial and final relative orbits (a_0, d_0, c_0) and (a_f, d_f, c_f) and the initial phase of the chaser, $0 \le \alpha_0 < 2\pi$, at t_0 are given. Then the chaser lies on the drifting ellipse with parameter (a_0, d_0, c_0) and the free motion of the chaser is given by

$$\begin{aligned} x(t) &= 2c_0 + a_0\cos[n(t - t_0) + \alpha_0], \\ y(t) &= d_0 - 3nc_0(t - t_0) - 2a_0\sin[n(t - t_0) + \alpha_0], \\ \dot{x}(t) &= -a_0 n\sin[n(t - t_0) + \alpha_0], \\ \dot{y}(t) &= -3nc_0 - 2a_0 n\cos[n(t - t_0) + \alpha_0] \end{aligned} \tag{15}$$

Let $m \in N$ be arbitrary, and consider the set of m instants $t_0 \le t_j < t_{j+1}$, $j = 1, 2, \ldots, m$ and m impulses $\Delta\dot{\mathbf{x}}_j$ at t_j. The m-impulse strategy $\Delta\dot{\mathbf{x}} = \{\Delta\dot{\mathbf{x}}_j; t_j\}$ is said to be admissible if the chaser maintains the final drifting (a_f, d_f, c_f) orbit after time t_m, that is, if the state of the chaser after the mth impulse satisfies

$$\begin{aligned} x(t) &= 2c_f + a_f\cos[n(t - t_m) + \alpha_f], \\ y(t) &= d_f - 3nc_f(t_m - t_0) - 3nc_f(t - t_m) \\ &\quad - 2a_f\sin[n(t - t_m) + \alpha_f], \\ \dot{x}(t) &= -a_f n\sin[n(t - t_m) + \alpha_f] \\ \dot{y}(t) &= -3nc_f - 2a_f n\cos[n(t - t_m) + \alpha_f] \end{aligned} \tag{16}$$

for some $0 \le \alpha_f < 2\pi$. Note that the deviation parameter of the final orbit becomes $d_f - 3nc_f(t_m - t_0)$ at t_m. The performance of this strategy is evaluated by the total change of velocity

$$\Delta V_T(\Delta\dot{\mathbf{x}}) = \sum_{j=1}^{m} (\Delta\dot{x}_j^2 + \Delta\dot{y}_j^2)^{1/2} \tag{17}$$

The minimum-fuel problem is to find the minimum (or infimum) of $\Delta V_T(\Delta\dot{\mathbf{x}})$ with respect to all admissible m-impulse strategies, $m \in N$, and to construct minimizing (or suboptimal) strategies. Thus, for our problem, the initial conditions are fixed, the final time is free, the number of impulses are arbitrary, and the position of the chaser on the final relative orbit is free. It is an open-time problem, and it has not been studied in the literature. Our formulation is, in principle, useful for those applications that allow long flight times.

If $a_0 > a_f > 0$, $d_0 > d_f \ge 0$, and $c_0 = c_f = 0$, the transfer could be interpreted as one from a temporary relative orbit to a relative orbit for inspection. The orbit transfer from (a_0, d_0, c_0) to $(a_f, d_f, 0)$ covers an approaching maneuver to a rendezvous orbit, whereas the transfer from $(a_0, d_0, 0)$ to (a_f, d_f, c_f) covers an escaping maneuver from a rendezvous orbit. Finally, the general transfer includes a transfer from a drifting orbit to a drifting flyaround orbit.

For ease of notation, the argument $\Delta\dot{\mathbf{x}}$ of ΔV_T will occasionally be omitted.

B. Out-of-Plane Motion

Now consider the state equation of the out-of-plane motion with impulse input given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -n^2 & 0 \end{bmatrix}\mathbf{x} = A_2\mathbf{x} \qquad (t \neq t_j),$$
$$\mathbf{x}(t_0) = [z_0 \quad \dot{z}_0]^T, \qquad \Delta\mathbf{x}(t_j) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Delta\dot{z}_j = B_2\Delta\dot{z}_j \tag{18}$$

where $\mathbf{x} = [z \quad \dot{z}]^T$. The free motion is given by

$$z(t) = z_0 \cos nt + \dot{z}_0 \sin nt/n, \qquad \dot{z}(t) = -nz_0 \sin nt + \dot{z}_0 \cos nt$$

Therefore, the out-of-plane motion is always periodic and is parameterized by b and β as

$$z(t) = b\cos(nt + \beta), \qquad \dot{z}(t) = -bn\sin(nt + \beta) \tag{19}$$

where

$$b = [z_0^2 + (\dot{z}_0/n)^2]^{1/2}, \qquad \cos\beta = z_0/b, \qquad \sin\beta = -\dot{z}_0/(bn) \tag{20}$$

In this case, it is convenient to consider the trajectory in the phase plane. It satisfies

$$\frac{z^2}{b^2} + \frac{\dot{z}}{(bn)^2} = 1 \tag{21}$$

and forms an ellipse with eccentricity $e = (1 - n^2)^{1/2}$ (see Fig. 2).

The parameter b represents the size of the ellipse, which is the maximum distance of the chaser from the orbit plane. The parameter β indicates the initial position of the chaser in the phase plane. The ellipse (21) will be referred to as an orbit b, and the parameters b and β are the size parameter and the initial phase, respectively. Let b_0 and b_f be the initial and final orbits, and consider the m-impulse transfer problem. Let $\Delta\dot{z} = \{\Delta\dot{z}_j; t_j\}$ be an admissible m-impulse strategy with total velocity change

$$\Delta V_T(\Delta\dot{z}) = \sum_{j=1}^{m} |\Delta\dot{z}_j| \tag{22}$$

Our minimum-fuel problem is to minimize ΔV_T over all admissible m-impulse strategies, $m \in N$, and to find minimizing strategies.

III. Lower Bounds of $\Delta V_T(\Delta\dot{\mathbf{x}})$ and Minimum of $\Delta V_T(\Delta\dot{z})$

In this section, two lower bounds of $\Delta V_T(\Delta\dot{\mathbf{x}})$ for the general transfer from (a_0, d_0, c_0) to (a_f, d_f, c_f) will be established, where $\Delta\dot{\mathbf{x}}$ is any admissible m-impulse strategy. In the next section, the minimum-fuel problem will be solved by finding control strategies attaining the lower bounds. In the case of the out-of-plane motion, the minimum of $\Delta V_T(\Delta\dot{z})$ for the transfer from b_0 to b_f is directly obtained.

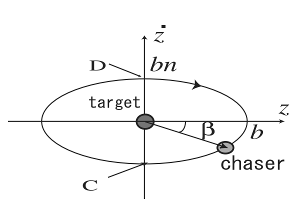


Fig. 2 Elliptic orbit in the phase plane.

A. Lower Bounds of V_T for In-Plane Motion

Consider the orbit transfer from (a_0, d_0, c_0) to (a_f, d_f, c_f), and let $\Delta\dot{\mathbf{x}}$ be an admissible m-impulse strategy. Then, the following lower bound holds.

Theorem 1.

$$\Delta V_T(\Delta\dot{\mathbf{x}}) \geq \max(n|a_f - a_0|/2, n|c_f - c_0|) \tag{23}$$

If, in particular, $c_0 = c_f = 0$,

$$\Delta V_T(\Delta\dot{\mathbf{x}}) \geq n|a_f - a_0|/2 \tag{24}$$

Note that the lower bound is independent of the initial phase α_0. The proof follows from the following two lemmas.

Lemma 1.

$$\Delta V_T(\Delta\dot{\mathbf{x}}) \geq n|a_f - a_0|/2 \tag{25}$$

Lemma 2.

$$\Delta V_T(\Delta\dot{\mathbf{x}}) \geq n|c_f - c_0| \tag{26}$$

Proof of Lemma 1. First consider the single-impulse transfer from (a, d, c) to $(a', *, *)$, where $*$ is arbitrary. Let the initial phase of the chaser at time t_0 be α. Consider an impulse $\Delta\dot{\mathbf{x}}_\tau = [\Delta\dot{x} \quad \Delta\dot{y}]^T$ at $\tau \geq t_0$. Then, from Eq. (11), the parameter a at τ becomes

$$a_1 = [[3x(\tau) + 2(\dot{y}(\tau) + \Delta\dot{y})/n]^2 + [(\dot{x}(\tau) + \Delta\dot{x})/n]^2]^{1/2} \tag{27}$$

Substitute Eq. (10) with $t = \tau$ into Eq. (27) to obtain

$$(\Delta\dot{x} - an\sin[n(\tau - t_0) + \alpha])^2$$
$$+ 4(\Delta\dot{y} - an\cos[n(\tau - t_0) + \alpha]/2)^2 = a_1^2 n^2 \tag{28}$$

Let $\bar{\alpha}$ be the phase at τ so that $n(\tau - t_0) + \alpha = \bar{\alpha} + 2k\pi$, $0 \leq \bar{\alpha} < 2\pi$, $k \in Z_+$. If $a_1 = a'$, then Eq. (28) becomes

$$\left(\frac{\Delta\dot{x} - an\sin\bar{\alpha}}{a'n}\right)^2 + \left(\frac{\Delta\dot{y} - (an/2)\cos\bar{\alpha}}{a'n/2}\right)^2 = 1 \tag{29}$$

Because Eq. (29) for $(\Delta\dot{x}, \Delta\dot{y})$ is an ellipse, it is parameterized as

$$\Delta\dot{x} = an\sin\bar{\alpha} + a'n\sin\delta,$$
$$\Delta\dot{y} = (an/2)\cos\bar{\alpha} + (a'n/2)\cos\delta \tag{30}$$

where $0 \leq \delta < 2\pi$. The parameter $\bar{\alpha}$ determines the position of the chaser at τ on the initial orbit, whereas δ is a free parameter that determines the impulse vector. The velocity change for this single impulse is given by

$$\Delta V_T(\Delta\dot{\mathbf{x}}_\tau) = (n/2)[a^2 + a'^2 + 3(a\sin\bar{\alpha} + a'\sin\delta)^2$$
$$+ 2aa'\cos(\bar{\alpha} - \delta)]^{1/2} \tag{31}$$

The optimal pair that minimizes Eq. (31) is $(\bar{\alpha}^*, \delta^*) = (0, \pi)$ or $(\pi, 0)$. In this case, Eq. (30) gives

$$\Delta\dot{x}^* = 0, \qquad \Delta\dot{y}^* = n(a - a')/2, \quad \text{when } \bar{\alpha}^* = 0;$$
$$\Delta\dot{x}^* = 0, \qquad \Delta\dot{y}^* = -n(a - a')/2, \quad \text{when } \bar{\alpha}^* = \pi \tag{32}$$

and the minimum velocity change is

$$\Delta V_T^* = n|a' - a|/2 \tag{33}$$

Note that ΔV_T^* is independent of the initial phase α. The optimal impulse time τ^* is parameterized by $k \in Z^+$ as

$$\tau^* = t_0 + (\bar{\alpha}^* + 2k\pi - \alpha)/n = t_0 + (\bar{\alpha}^* - \alpha)/n + kT \tag{34}$$

In view of Eq. (10),

$$\dot{x}(\tau^*) = -an\sin[n(\tau^* - t_0) + \alpha] = -an\sin\bar{\alpha}^* = 0 \tag{35}$$

If $c = 0$, the position of the chaser at τ^* is either point A or B in Fig. 1, where $\dot{x}(\tau^*) = 0$. In fact, $\bar{\alpha}^* = 0$ corresponds to A, whereas $\bar{\alpha}^* = \pi$ corresponds to B. In the inertial frame, these points correspond approximately to the apogee ($\bar{\alpha}^* = 0$) and the perigee ($\bar{\alpha}^* = \pi$) of the orbit of the chaser. If $\alpha = \bar{\alpha}^* = 0$ and $k = 0$ are selected, then $\tau^* = t_0$ and the impulse is applied immediately and the transfer time is 0. Note that the velocity change is along the y direction and is independent of parameters d and c.

Now consider the transfer from (a_0, d_0, c_0) to $(a_f, *, *)$ by m impulses. Let $\Delta\dot{\mathbf{x}} = \{\Delta\dot{\mathbf{x}}_j; t_j\}$ be an m-impulse strategy with $t_0 \le t_1 < \cdots < t_m$ such that $a_m = a_f$, where a_j is the value of the parameter a after the jth impulse at t_j. In view of Eq. (33), the total velocity change of this strategy satisfies

$$\Delta V_T(\Delta\dot{\mathbf{x}}) = \sum_{j=1}^{m} \Delta V_T(\Delta\dot{\mathbf{x}}_j) \ge \sum_{j=1}^{m} n|a_j - a_{j-1}|/2 \ge n|a_f - a_0|/2 \tag{36}$$

where equalities hold if and only if a_j $(j = 1, \ldots, m-1)$ satisfy the monotonicity condition

$$a_j < a_{j+1} \quad (\text{if } a_0 < a_f), \quad \text{or} \quad a_j > a_{j+1} \quad (\text{if } a_0 > a_f), \quad j = 0, \ldots, m-1 \tag{37}$$

and the pair $(t_j, \Delta\dot{\mathbf{x}}_j^*)$ is chosen such that

$$\Delta V_T(\Delta\dot{\mathbf{x}}_j^*) = n|a_j - a_{j-1}|/2 \tag{38}$$

In fact, for the strategy $\Delta\dot{\mathbf{x}}^* = \{\Delta\dot{\mathbf{x}}_j^*; t_j\}$,

$$\Delta V_T(\Delta\dot{\mathbf{x}}^*) = n|a_f - a_0|/2$$

Note that the lower bound is independent of the number of impulses. This gives a lower bound on the total velocity change required for any transfer from (a_0, d_0, c_0) to (a_f, d_f, c_f). This completes the proof of Lemma 1.

After the impulse (32) at τ^*, parameters d, c, and α can be evaluated by Eq. (11) and become

$$\begin{aligned} d' &= y(\tau^*) - 2[\dot{x}(\tau^*) + \Delta\dot{x}^*]/n = d - 3nc(\tau^* - t_0), \\ c' &= 2x(\tau^*) + (\dot{y}(\tau^*) + \Delta\dot{y}^*)/n = c + \Delta\dot{y}^*/n, \\ \cos\alpha' &= -(1/a')(3x(\tau^*) + 2[\dot{y}(\tau^*) + \Delta\dot{y}^*]/n) \\ &= (a/a')\cos\bar{\alpha}^* - (2/na')\Delta\dot{y}^*, \\ \sin\alpha' &= -(1/na')[\dot{x}(\tau^*) + \Delta\dot{x}^*] = (a/a')\sin\bar{\alpha}^* = 0 \end{aligned}$$

where Eq. (10) with $t = \tau^*$ is used. Substitute Eq. (32) into these equations to obtain

$$\begin{aligned} &d' = d - 3nc(\tau^* - t_0), \qquad c' = c + (a - a')/2, \\ &\alpha' = 0 \quad \text{when } \bar{\alpha}^* = 0, \qquad d' = d - 3nc(\tau^* - t_0), \\ &c' = c - (a - a')/2, \qquad \alpha' = \pi \quad \text{when } \bar{\alpha}^* = \pi \end{aligned} \tag{39}$$

The phase α' after the impulse obviously remains the same as $\bar{\alpha}^*$. The parameter k in Eq. (34) is useful when optimal three-impulse transfers are established in the next section.

Proof of Lemma 2. Consider a single-impulse transfer from (a, d, c) to $(*, *, c')$ by an impulse $\Delta\dot{\mathbf{x}}_\tau = [\,\Delta\dot{x} \quad \Delta\dot{y}\,]^T$ at τ. From Eq. (11), c at τ becomes

$$c_1 = 2x(\tau) + [\dot{y}(\tau) + \Delta\dot{y}]/n \tag{40}$$

Substitute Eq. (10) with $t = \tau$ into Eq. (40) to obtain

$$c_1 = c + \Delta\dot{y}/n \tag{41}$$

If $c_1 = c'$, then $\Delta\dot{y} = n(c' - c)$, and

$$\Delta V_T(\Delta\dot{\mathbf{x}}_\tau) = [\Delta\dot{x}^2 + n^2(c' - c)^2]^{1/2} \tag{42}$$

The impulse $(\Delta\dot{x}, \Delta\dot{y})$, which minimizes Eq. (42), is

$$\Delta\dot{x}^* = 0, \qquad \Delta\dot{y}^* = n(c' - c) \tag{43}$$

and the minimum velocity change is

$$\Delta V_T^* = n|c' - c| \tag{44}$$

Note that the optimal impulse is along the y direction and is independent of the parameters a and d. Note also that the impulse time τ is free.

Now consider the orbit transfer from (a_0, d_0, c_0) to $(*, *, c_f)$ by m impulses $\Delta\dot{\mathbf{x}} = \{\Delta\dot{\mathbf{x}}_j; t_j\}$. The same reasoning based on the monotonicity condition used for Lemma 1 then shows that the minimum total velocity change is independent of m, and

$$\min \Delta V_T(\Delta\dot{\mathbf{x}}) = n|c_f - c_0| \tag{45}$$

This gives the second lower bound for $\Delta V_T(\Delta\dot{\mathbf{x}})$ of the transfer from (a_0, d_0, c_0) to (a_f, d_f, c_f), which completes the proof of Lemma 2.

B. Minimum for Out-of-Plane Motion

Let the initial phase of the chaser at t_0 be β_0, and consider the transfer from b_0 to b_f. The solution to the minimum-fuel problem is given by the following theorem.

Theorem 2. The single-impulse strategy $\{\Delta\dot{z}^*, t^*\}$ given here is optimal with minimum velocity change $\Delta V_T(\Delta\dot{z}^*) = n|b_f - b_0|$, where

$$t^* = t_0 + (\beta^* - \beta_0)/n + kT, \qquad k \in Z^+ \tag{46}$$

$\bar{\beta}^* = \pi/2$ or $3\pi/2$, and

$$\begin{aligned} \Delta\dot{z}^* &= n(b_0 - b_f) \quad \text{when } \beta^* = \pi/2, \\ \Delta\dot{z}^* &= -n(b_0 - b_f) \quad \text{when } \beta^* = 3\pi/2 \end{aligned}$$

Proof. Let the initial phase of the chaser at t_0 be β, and consider the transfer from b to b'. Now examine how the size parameter b changes after an impulse $\Delta\dot{z}$ at t. From Eq. (20), the parameter b becomes

$$b_1 = [z^2(t) + (\dot{z}(t) + \Delta\dot{z})^2/n^2]^{1/2} \tag{47}$$

Substitute Eq. (21) into Eq. (47) to obtain

$$\Delta\dot{z}^2 - 2bn\sin[n(t - t_0) + \beta]\Delta\dot{z} + (b^2 - b_1^2)n^2 = 0 \tag{48}$$

Setting $b_1 = b'$ in Eq. (48) and solving it for $\Delta\dot{z}$, one arrives at

$$\Delta\dot{z} = bn\sin\bar{\beta} \pm n(b'^2 - b^2\cos^2\bar{\beta})^{1/2} \tag{49}$$

where $n(t - t_0) + \beta = \bar{\beta} + 2k\pi$ $(0 \le \bar{\beta} < 2\pi,\ k \in Z_+)$. Substitution of Eq. (49) into Eq. (22) with $m = 1$ yields

$$\Delta V_T(\Delta\dot{z}) = n|b\sin\bar{\beta} \pm (b'^2 - b^2\cos^2\bar{\beta})^{1/2}|$$

This is minimized by $\bar{\beta}^* = \pi/2$ or $3\pi/2$, and the optimal impulse is

$$\begin{aligned} \Delta\dot{z}^* &= n(b - b') \quad \text{when } \bar{\beta}^* = \pi/2, \\ \Delta\dot{z}^* &= -n(b - b') \quad \text{when } \bar{\beta}^* = 3\pi/2 \end{aligned} \tag{50}$$

and the total velocity change is

$$\Delta V_T(\Delta\dot{z}^*) = n|b' - b| \tag{51}$$

The proof follows by setting $b = b_0$ and $b' = b_f$.

The optimal phase $\bar{\beta}^*$ determines the position of the chaser at the impulse time $t = t^*$. Because

$$z(t^*) = b\cos[n(t^* - t_0) + \beta] = b\cos\bar{\beta}^* = 0$$

t^* is the time at which the chaser intersects with the orbit plane. Points C and D in Fig. 2 are those points.

Note that optimal m-impulse strategies can be constructed by taking increasing or decreasing intermediate orbits and applying optimal single impulses successively. A multi-impulse transfer is useful for a docking problem to reduce the final velocity change.

Remark 1. In view of Eqs. (33) and (51), the minimum velocity change required to change the size parameter for the out-of-plane motion is lager, by a factor of two, than that for the in-plane motion.

IV. Optimal and Suboptimal Strategies for In-Plane Motion

Let (a_0, d_0, c_0) and (a_f, d_f, c_f) be the initial and final orbits for the in-plane motion, respectively. The goal of this section is to establish

$$\inf \Delta V_T(\Delta\dot{\mathbf{x}}) = \max(n|a_f - a_0|/2, n|c_f - c_0|)$$

and to show that, at most, three impulses are necessary to attain the minimum or to make the total velocity change arbitrarily close to the infimum.

In view of Theorem 1, the lower bound of $\Delta V_T(\Delta\dot{\mathbf{x}})$ is $n|a_f - a_0|/2$ if $|a_f - a_0|/2 > |c_f - c_0|$. In this case, optimal velocity changes with respect to a are adopted as candidates for optimal strategies. In fact, if the requirements for d and c are satisfied, then the lower bound is achieved and impulse strategies become optimal. If, on the other hand, $|a_f - a_0|/2 < |c_f - c_0|$, the lower bound is $n|c_f - c_0|$. In this case, optimal velocity changes with respect to c are taken as candidates, and the requirements for a and d are examined. Let the initial phase at t_0 be α_0.

A. Case 1: $|a_f - a_0|/2 > |c_f - c_0|$

In this case, optimal three-impulse strategies, which attain the lower bound, will be constructed. Note that by assumption $a_f \neq a_0$. Our main results are summarized in the following theorem.

Theorem 3. Suppose $|a_f - a_0|/2 > |c_f - c_0|$ holds. Then the minimal total velocity change is $\Delta V_T^* = n|a_f - a_0|/2$ and the three-impulse strategy $\Delta\dot{\mathbf{x}}^* = \{\Delta\dot{\mathbf{x}}_j^*; t_j\}$ given by

$$\Delta\dot{\mathbf{x}}_j^* = [0 \quad \Delta\dot{y}_j^*]^T, \qquad n(t_j - t_{j-1}) + \alpha_{j-1}^* = \bar{\alpha}_j^* + 2k_j\pi,$$
$$\Delta\dot{y}_j^* = \begin{cases} n(a_{j-1} - a_j)/2 & \text{if } \bar{\alpha}_j^* = 0 \\ -n(a_{j-1} - a_j)/2 & \text{if } \bar{\alpha}_j^* = \pi \end{cases} \tag{52}$$

with $\bar{\alpha}_0^* = \alpha_0$ and $a_3 = a_f$ is well defined and optimal if 1) $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, \pi)$, 2) a_j, $j = 1, 2$ are given by

$$\begin{aligned} a_1 &= a_0 + [(a_f - a_0)/2 - (c_f - c_0)](k_3 + 1/2)/(k_2 + k_3) \\ &\quad + [(d_0 - d_f)/3\pi + (c_f - c_0) \\ &\quad \times (2k_1 + 2k_2 + 2k_3 + 1 - \alpha_0/\pi)]/(k_2 + k_3), \\ a_2 &= a_f - 2(c_f - c_0) - [(a_f - a_0)/2 - (c_f - c_0)] \\ &\quad \times (k_2 - 1/2)/(k_2 + k_3) + [(d_0 - d_f)/3\pi \\ &\quad + (c_f - c_0)(2k_1 + 2k_2 + 2k_3 + 1 - \alpha_0/\pi)]/(k_2 + k_3) \end{aligned} \tag{53}$$

and 3) $k_1, k_3 \in Z_+$, $k_2 \in N$ are any integers satisfying, respectively, feasible inequalities

$$2(c_f - c_0)[2(k_1 + k_2)\pi - \alpha_0] - [(a_0 - a_f)/2 - (c_f - c_0)](2k_3 + 1)\pi < (2/3)(d_f - d_0) \tag{54}$$

$$2(c_f - c_0)[(2k_1 + 1)\pi - \alpha_0] + [(a_0 - a_f)/2 + c_f - c_0](2k_2 - 1)\pi > (2/3)(d_f - d_0) \tag{55}$$

when $a_0 > a_f$, or feasible inequalities

$$2(c_f - c_0)[2(k_1 + k_2)\pi - \alpha_0)] + [(a_f - a_0)/2 + c_f - c_0](2k_3 + 1)\pi > (2/3)(d_f - d_0) \tag{56}$$

$$2(c_f - c_0)[(2k_1 + 1)\pi - \alpha_0] - [(a_f - a_0)/2 - (c_f - c_0)](2k_2 - 1)\pi < (2/3)(d_f - d_0) \tag{57}$$

when $a_0 < a_f$.

Proof. To show that the three-impulse strategy (52) is well defined, a_j will be determined first. In view of the proof of Lemma 1, the impulse $\Delta\dot{\mathbf{x}}_j^*$ at t_j changes parameter a from a_{j-1} to a_j with minimum velocity change $n|a_j - a_{j-1}|/2$. To construct optimal three-impulse strategies, parameters k_j and $\bar{\alpha}_j^*$ will be selected such that the monotonicity condition on a_j and the requirements $c_3 = c_f$ and $d_3 = d_f$ are satisfied. After each impulse, c_j and d_j are given by Eq. (39) as follows:

$$\begin{aligned} (c_1, \bar{\alpha}_1^*) &= \left(c_0 + \frac{a_0 - a_1}{2}, 0\right) \quad \text{or} \quad \left(c_0 - \frac{a_0 - a_1}{2}, \pi\right) \\ (c_2, \bar{\alpha}_2^*) &= \left(c_1 + \frac{a_1 - a_2}{2}, 0\right) \quad \text{or} \quad \left(c_1 - \frac{a_1 - a_2}{2}, \pi\right) \\ (c_3, \bar{\alpha}_3^*) &= \left(c_2 + \frac{a_2 - a_f}{2}, 0\right) \quad \text{or} \quad \left(c_2 - \frac{a_2 - a_f}{2}, \pi\right) \\ d_1 = d_0 - 3nc_0\Delta t_1, &\quad d_2 = d_1 - 3nc_1\Delta t_2, \quad d_3 = d_2 - 3nc_2\Delta t_3 \end{aligned} \tag{58}$$

where $\Delta t_j = t_j - t_{j-1}$ and

$$(c_1, \bar{\alpha}_1^*) = (c_0 + \frac{a_0 - a_1}{2}, 0)$$

for example, indicates $c_1 = c_0 + (a_0 - a_1)/2$ if $\bar{\alpha}_1^* = 0$. In fact, by Eq. (39), $d_j = d_{j-1} - 3nc_{j-1}\Delta t_j$ and $c_j = c_{j-1} + (a_{j-1} - a_j)/2$ (if $\alpha_j^* = 0$) after the impulse at t_j. For orbit transfer, the equalities $d_3 = d_f - 3nc_f(\Delta t_1 + \Delta t_2 + \Delta t_3)$ and $c_3 = c_f$ are required. Now set $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, \pi)$. Then,

$$\begin{aligned} \Delta t_1 &= (\bar{\alpha}_1^* + 2k_1\pi - \alpha_0)n = [(2k_1 + 1)\pi - \alpha_0]/n, \\ \Delta t_2 &= (\bar{\alpha}_2^* + 2k_2\pi - \bar{\alpha}_1^*)/n = (2k_2 - 1)\pi/n, \\ \Delta t_3 &= (\bar{\alpha}_3^* + 2k_3\pi - \bar{\alpha}_2^*)/n = (2k_3 + 1)\pi/n \end{aligned} \tag{59}$$

where $k_1, k_3 \in Z^+$, and $k_2 \in N$. The required equalities, $d_3 = d_f$ and $c_3 = c_f$, and Eq. (58) yield equations for a_j, $j = 1, 2$:

$$\begin{aligned} d_f &= d_0 + 3(c_f - c_0)[(2k_1 + 1)\pi - \alpha_0] \\ &\quad + 3(c_f - c_1)(2k_2 - 1)\pi + 3(c_f - c_2)(2k_3 + 1)\pi, \\ c_f &= c_0 - (a_0 - 2a_1 + 2a_2 - a_f)/2, \\ c_1 &= c_0 - (a_0 - a_1)/2, \qquad c_2 = c_1 + (a_1 - a_2)/2 \end{aligned} \tag{60}$$

These equations are solved for a_1 and a_2, and Eq. (53) is obtained.

To show that $\Delta\mathbf{x}^* = \{\Delta\dot{\mathbf{x}}_j^*, t_j\}$ for the suitable choice of k_j is optimal, only the monotonicity condition should be assured. First, consider the case $a_0 > a_f$. Then the monotonicity condition becomes $a_0 > a_1 > a_2 > a_f$. The condition $a_0 > a_1$ is equivalent to the inequality (54). The condition $a_1 > a_2$ is satisfied because $(a_0 - a_f)/2 + c_f - c_0 > 0$. The condition $a_2 > a_f$ is equivalent to Eq. (55). Because $(a_0 - a_f)/2 > |c_f - c_0|$, for any given $k_1 \in Z^+$, Eq. (55) is satisfied for large k_2 and, for this pair (k_1, k_2), Eq. (54) holds for large k_3. Thus, the monotonicity condition is satisfied and the orbit transfer is fulfilled with minimum total velocity change $n(a_0 - a_f)/2$.

Now consider the case $a_0 < a_f$. Then the monotonicity condition is $a_0 < a_1 < a_2 < a_f$, and inequalities (54) and (55) are replaced by inequalities (56) and (57), respectively. Because $(a_f - a_0)/2 > |c_f - c_0|$, parameters k_j, $j = 1$–3 can be chosen such that inequalities (56) and (57) hold. Thus, the monotonicity condition is satisfied, and the three-impulse strategy $\Delta\dot{\mathbf{x}} = \{\Delta\dot{\mathbf{x}}_j^*; t_j\}$ is optimal, completing the proof.

$(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, \pi)$ is not the only choice for optimality. For $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, \pi, 0)$, optimal three-impulse strategies can be constructed in a similar manner. Note that the minimum velocity change V_T^* is independent of t_0 and α.

In the special case $c_0 = c_f = 0$, inequalities in the theorem are simplified, and the following corollary is obtained. In this case, the two relative orbits are periodic and elliptic, which are particularly important for passive flyaround and formation flight.

Corollary 1. Consider the transfer from $(a_0, d_0, 0)$ to $(a_f, d_f, 0)$. Define k_j as

$$\begin{cases} k_1 \geq 0 & \text{if } 0 \leq \alpha_0 \leq \pi \\ k_1 \geq 1 & \text{if } \pi < \alpha_0 < 2\pi \\ k_2 \geq 1, k^* - 1 < k_3 \in N, & \text{if } (a_0 - a_f)(d_0 - d_f) > 0 \\ k^* < k_2 \in N, k_3 \geq 0, & \text{if } (a_0 - a_f)(d_0 - d_f) < 0 \\ k_2 \geq 1, k_3 \geq 0, & \text{if } d_0 = d_f \end{cases} \tag{61}$$

and a_j, $j = 1, 2$ by

$$\begin{aligned} a_1 &= a_0 + [(a_f - a_0)/2](k_3 + 1/2)/(k_2 + k_3) \\ &\quad + (d_0 - d_f)/[3\pi(k_2 + k_3)] \\ a_2 &= a_f - [(a_f - a_0)/2](k_2 - 1/2)/(k_2 + k_3) \\ &\quad + (d_0 - d_f)/[3\pi(k_2 + k_3)] \end{aligned} \tag{62}$$

where

$$k^* = (1/2)[1 + 4|d_0 - d_f|/3|a_0 - a_f|\pi] \tag{63}$$

is a positive number. Then the three-impulse strategy (52) is optimal, and the minimum total velocity change is $\Delta V_T^* = n|a_f - a_0|/2$.

Proof. If $c_0 = c_f = 0$, Eq. (53) is simplified to Eq. (62). First, assume $a_0 > a_f$. Then Eq. (54) becomes

$$k_3 > (1/2)[1 + 4(d_0 - d_f)/3n(a_0 - a_f)\pi] - 1$$

whereas Eq. (55) yields

$$k_2 > (1/2)[(1 - 4(d_0 - d_f)/3n(a_0 - a_f)\pi]$$

If $d_0 \geq d_f$, choose $k_2 \geq 1$ and $k^* - 1 < k_3 \in N$. On the other hand, if $d_0 < d_f$, choose $k^* < k_2 \in N$ and $k_3 \geq 0$. Then in each case the monotonicity condition is satisfied, and the transfer from $(a_0, d_0, 0)$ to $(a_f, d_f, 0)$ with minimum total velocity change $\Delta V_T^* = \frac{n}{2}|a_f - a_0|$ is fulfilled.

Now assume $a_0 < a_f$. In this case, Eq. (56) leads to

$$k_3 > (1/2)[1 - 4(d_0 - d_f)/3(a_f - a_0)\pi] - 1$$

whereas Eq. (57) is equivalent to

$$k_2 > (1/2)[1 + 4(d_0 - d_f)/3(a_f - a_0)\pi]$$

If $d_0 \geq d_f$, choose $k^* < k_2 \in N$ and $k_3 \geq 0$. On the other hand, if $d_0 < d_f$, choose $k_2 \geq 1$ and $k^* < k_3 \in N$. These conditions are summarized in a compact manner as Eq. (61).

Note that the transfer time $t_f = \Delta t_1 + \Delta t_2 + \Delta t_3$ can be minimized by the choice of smallest admissible k_j. The three-impulse transfer is described in Fig. 3. The initial relative orbit is the ellipse denoted by $E_0(a_0)$ with size parameter a_0, and the initial position of the chaser is at point A. The chaser stays in the initial orbit until it reaches point B. There, the first impulse is applied, and the drifting ellipse on which the chaser stays after the impulse is denoted by $E_1(a_1)$. It drifts to $E_1'(a_1)$ and, at point C, the second impulse is applied. The chaser then gets on the smaller drifting ellipse $E_2(a_2)$. It drifts to $E_2'(a_2)$, and the third impulse is applied to the chaser at point D, which brings the chaser to the final orbit $E_f(a_f)$.

Remark 2. If $c_f = 0$, a single-impulse transfer is not possible because $c_1 \neq 0$ for $a_0 \neq a_1 = a_f$. A two-impulse transfer is possible only for special cases. For example, the final orbit should coincides with some $E_2(a_2)$, which is not drifting.

Remark 3. For the docking problem, that is, $(a_f, d_f, c_f) = (0, 0, 0)$, the final velocity change is important. Using Eqs. (52) and (62), it is given by

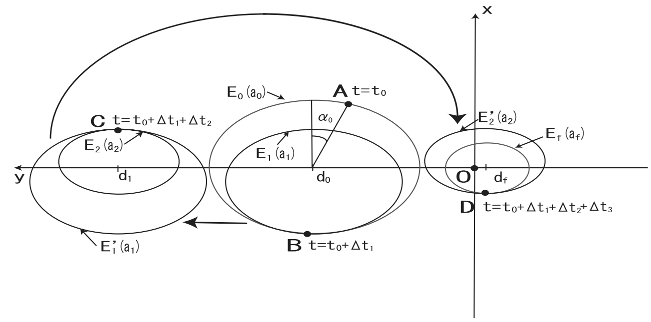


Fig. 3 Three-impulse transfer.

$$\Delta \dot{y}_3^* = -na_2/2 = -[d_0/3 + (2k_2 - 1)a_0/4]/(k_2 + k_3 - 1)$$

Hence, selecting a large k_3, $\Delta \dot{y}_3^*$ can be made arbitrarily small.

Remark 4. The optimal velocity changes in Theorem 2 are along the y direction. Taking only velocity changes $\Delta \dot{x}$, the minimum-fuel problem for a single-impulse transfer from (a, d, c) to $(a', *, *)$ can be solved. In view of Eq. (30), optimal pairs are $(\bar{\alpha}, \delta) = (\pi/2, 3\pi/2)$ and $(3\pi/2, \pi/2)$ with optimal velocity change

$$\begin{aligned} \Delta \dot{x} &= n(a - a'), \qquad \Delta \dot{y} = 0 \quad \text{when } \bar{\alpha} = \pi/2, \\ \Delta \dot{x} &= -n(a - a'), \qquad \Delta \dot{y} = 0 \quad \text{when } \bar{\alpha} = 3\pi/2 \end{aligned} \tag{64}$$

The minimum velocity change is $\Delta V_T^* = n|a' - a|$. Similarly, considering the minimum-fuel problem for the transfer from (a, d, c) to $(*, d', *)$, the following lemma can be proved.

Lemma 3. Consider the transfer from (a_0, d_0, c_0) to (a_f, d_f, c_f) by velocity changes $\Delta \dot{x}$. Then the following lower bound holds:

$$\Delta V_T(\Delta \dot{x}) \geq \max(n|a_f - a_0|, n|d_f - d_0|/2) \tag{65}$$

for any admissible m-impulse strategy $\Delta \dot{x}$.

It can be shown that the lower bound is attained by at most two impulses.

B. Case 2: $|a_f - a_0|/2 < |c_f - c_0|$

Note that $c_f \neq c_0$ in view of the assumption. In this case, the lower bound of the total velocity change is $n|c_f - c_0|$. Here impulses $\Delta \dot{\mathbf{x}}_j^* = [0 \quad \Delta \dot{y}_j^*]^T$ with $\Delta \dot{y}_j^* = n(c_j - c_{j-1})$ are employed. They are optimal in changing parameter c from c_{j-1} to c_j. Note that impulse times are not specified for this change. Here the same impulse times Δt_j given in case 1 are employed, that is, $n(t_j - t_{j-1}) + \bar{\alpha}_{j-1}^* = \bar{\alpha}_j^* + 2k_j\pi$, $j = 1$–3, where $\bar{\alpha}_j^* = 0$, or π, $j = 1$–3, and $\bar{\alpha}_0^* = \alpha_0$. This choice is natural in that the velocity change is tangential to the relative orbit as in case 1. It also justifies the choice of impulse times $jT/2$ when feedback controls are designed in the next section.

Contrary to case 1, optimal strategies, which attain the lower bound $n|c_f - c_0|$, do not exist in general. However, there exist suboptimal three-impulse strategies whose total velocity change is arbitrarily close to the lower bound. Our main results are summarized as follows:

1) Consider the case $c_0 > c_f$.

a) If $a_f > 0$, suboptimal strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, \pi, \pi)$ exist.

b) If $a_f = 0$, suboptimal strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, \pi, \pi)$ or $(0, \pi, 0)$ exist.

c) Optimal strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, 0, \pi)$ exist if there are $k_1, k_3 \in Z_+$, and $k_2 \in N$ that satisfy the inequalities

$$\begin{aligned} (d_0 - d_f)/3 &< (c_0 - c_f)[2(k_1 + k_2)\pi - \alpha_0] \\ &\quad + (1/2)[c_0 - c_f - (a_f - a_0)/2](2k_3 + 1)\pi \end{aligned} \tag{66}$$

$$(c_0 - c_f)(2k_1\pi - \alpha_0) + (1/2)[c_0 - c_f - (a_f - a_0)/2] \times [2(k_2 + k_3) + 1]\pi < (d_0 - d_f)/3 \tag{67}$$

2) Consider the case $c_0 < c_f$.

a) If $a_f > 0$, suboptimal strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, 0)$ exist.

b) If $a_f = 0$, suboptimal strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, 0)$ or $(\pi, 0, \pi)$ exist.

c) Optimal strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, \pi, 0)$ exist if there are $k_1, k_3 \in Z_+$, and $k_2 \in N$ that satisfy the inequalities

$$(d_f - d_0)/3 < (c_f - c_0)[(2(k_1 + k_2) + 1)\pi - \alpha_0] + (1/2)[c_f - c_0 - (a_f - a_0)/2](2k_3 - 1)\pi \tag{68}$$

$$(c_f - c_0)[(2k_1 + 1)\pi - \alpha_0] + (1/2)[c_f - c_0 - (a_f - a_0)/2] \times [2(k_2 + k_3) - 1]\pi < (d_f - d_0)/3 \tag{69}$$

Proof. After the impulse $\Delta\dot{\mathbf{x}}_j^*$, parameters a and d are given by

$$\begin{aligned}(a_1, \bar{\alpha}_1^*) &= (|a_0 - 2(c_1 - c_0)|, 0) \quad \text{or} \quad (|a_0 + 2(c_1 - c_0)|, \pi),\\ (a_2, \bar{\alpha}_2^*) &= (|a_1 - 2(c_2 - c_1)|, 0) \quad \text{or} \quad (|a_1 + 2(c_2 - c_1)|, \pi),\\ (a_3, \bar{\alpha}_3^*) &= (|a_2 - 2(c_f - c_2)|, 0) \quad \text{or} \quad (|a_2 + 2(c_f - c_2)|, \pi),\\ d_3 &= d_0 - 3nc_0\Delta t_1 - 3nc_1\Delta t_2 - 3nc_2\Delta t_3\end{aligned} \tag{70}$$

In fact, if, for example, $\bar{\alpha}_j^* = 0$, then by Eqs. (10) and (11)

$$\begin{aligned}a_j &= [(3x(t_j) + (2/n)(\dot{y}(t_j) + \Delta\dot{y}_j^*)^2 + (\dot{x}(t_j)/n)^2]^{1/2}\\ &= |3(2c_j + a_j\cos\bar{\alpha}_j^*) - 2(3c_j + 2a_j\cos\bar{\alpha}_j^*) + c_j - c_{j-1}|\\ &= |a_j - 2(c_j - c_{j-1})|\end{aligned}$$

For the orbit transfer, equalities $d_3 = d_f - 3nc_f(\Delta t_1 + \Delta t_2 + \Delta t_3)$ and $a_3 = a_f$ are required.

First, consider case 1c. Setting $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, 0, \pi)$, the parameters c_j, $j = 1, 2$ will be determined so that the monotonicity condition $c_0 > c_1 > c_2 > c_f$ is satisfied. In this case, Eq. (70) and the aforementioned requirements yield

$$\begin{aligned}a_1 &= |a_0 - 2(c_1 - c_0)| = a_0 + 2(c_0 - c_1),\\ a_2 &= |a_1 - 2(c_2 - c_1)| = a_0 + 2(c_0 - c_2)\end{aligned} \tag{71}$$

$$\begin{aligned}a_3 &= |a_2 + 2(c_f - c_2)| = |a_0 - 2(2c_2 - c_f - c_0)| = a_f,\\ d_f &= d_0 + 3n(c_f - c_0)\Delta t_1 + 3n(c_f - c_1)\Delta t_2 + 3n(c_f - c_2)\Delta t_3\end{aligned} \tag{72}$$

Now, in view of the assumption $|a_f - a_0|/2 < |c_f - c_0|$, c_2 given by

$$c_2 = (1/2)[c_f + c_0 - (a_f - a_0)/2] \tag{73}$$

is positive and satisfies Eq. (71) and the inequality $c_0 > c_2 > c_f$. Determine c_1 by Eq. (72). Then

$$c_1 = c_f - (1/6k_2\pi)[d_f - d_0 - 3(c_f - c_0)(2k_1\pi - \alpha_0) - 3(c_f - c_2)(2k_3 + 1)\pi] \tag{74}$$

which assures $d_3 = d_f$. The condition $c_0 > c_1$ is then equivalent to the inequality (66), and the condition $c_1 > c_2$ is equivalent to the inequality (67). Note that the inequalities (66) and (67) are consistent in the sense that the right-hand side of the inequality (66) is greater than the left-hand side of the inequality (67). If there exist integers k_1, $k_2 \in N$, and $k_3 \in Z_+$ that satisfy inequalities (66) and (67), the three-impulse strategy $\Delta\dot{\mathbf{x}}^* = \{\Delta\dot{\mathbf{x}}_j^*; t_j\}$ is optimal.

Now, consider the case 1a. Set $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, \pi, \pi)$ and choose

$$c_1 = (1/2)[c_f + c_0 - (a_f - a_0)/2] \tag{75}$$

Note that the assumption $|a_f - a_0|/2 < |c_f - c_0|$ implies $c_0 > c_1 > c_f$. Using Eq. (72), set $c_2 = c_f - \epsilon$, where

$$\epsilon = (1/3n\Delta t_3)[d_f - d_0 + 3n(c_0 - c_f)\Delta t_1 + 3n(c_1 - c_f)\Delta t_2] \tag{76}$$

Then $d_3 = d_f$. Choosing a large k_3, ϵ can be made arbitrarily small so that $a_f - 2\epsilon \geq 0$. Now, from Eq. (70)

$$\begin{aligned}a_1 &= |a_0 - 2(c_1 - c_0)| = a_0 + 2(c_0 - c_1),\\ a_2 &= |a_1 + 2(c_2 - c_1)| = |a_0 - 2(2c_1 - c_0 - c_2)| = |a_f - 2\epsilon|\\ &= a_f - 2\epsilon,\\ a_3 &= |a_2 + 2(c_f - c_2)| = |a_f + 2\epsilon - 2\epsilon| = a_f\end{aligned}$$

where Eq. (75) is used for the second equality. Because ϵ is arbitrarily small, $a_2 > 0$. Hence, the requirements on a_3 and d_3 are satisfied. The total velocity change is

$$\begin{aligned}\Delta V_T(\Delta\dot{\mathbf{x}}^*) &= n(|c_1 - c_0| + |c_2 - c_1| + |c_f - c_2|) = n(|c_0 - c_1|\\ &\quad + |c_f - \epsilon - c_1| + \epsilon)\\ &\leq n(|c_0 - c_1| + |c_1 - c_f| + 2\epsilon) = n(c_0 - c_f) + 2n\epsilon\end{aligned}$$

because $c_0 > c_1 > c_f$. Thus, ΔV_T can be made arbitrarily close to the lower bound $n|c_f - c_0|$.

Next, consider the case 1b. If ϵ is nonnegative, then a_2 is nonnegative and the same conclusion holds. If $\epsilon > 0$, keep c_1 and c_2 as they are and change $\bar{\alpha}_3^*$ to 0. Again, by Eq. (70)

$$\begin{aligned}a_2 &= |0 - 2\epsilon| = 2\epsilon,\\ a_3 &= |a_2 - 2(c_f - c_2)| = |2\epsilon - 2\epsilon| = 0 = a_f\end{aligned}$$

Hence, the conditions on a_3 and d_3 are satisfied, and

$$\Delta V_T(\Delta\dot{\mathbf{x}}^*) \leq n|c_f - c_0| + 2n|\epsilon|$$

Again, ΔV_T can be made arbitrarily close to the infimum $n|c_f - c_0|$.

Now, consider the case 2c. For the choice $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, \pi, 0)$, Eq. (70) and the requirements $a_3 = a_f$ and $d_3 = d_f$ yield

$$\begin{aligned}&a_1 = a_0 + 2(c_1 - c_0), \qquad a_2 = a_0 + 2(c_2 - c_0),\\ &a_3 = |a_2 - 2(c_f - c_2)| = |a_0 + 2(2c_2 - c_f - c_0)| = a_f\end{aligned} \tag{77}$$

and

$$d_f = d_0 + 3n(c_f - c_0)\Delta t_1 + 3n(c_f - c_1)\Delta t_2 + 3n(c_f - c_2)\Delta t_3 \tag{78}$$

Equation (77) gives

$$c_2 = (1/2)[c_f + c_0 + (a_f - a_0)/2] \tag{79}$$

and, hence, $c_0 < c_2 < c_f$, whereas Eq. (78) yields

$$c_1 = c_f - (1/3n\Delta t_2)[d_f - d_0 - 3n(c_f - c_0)\Delta t_1 - 3n(c_f - c_2)\Delta t_3] \tag{80}$$

As in the case 1c, the monotonicity condition leads to two inequalities (68) and (69).

Similarly, suboptimal three-impulse strategies can be constructed by setting $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, 0)$ in case 2a and $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, 0)$ or $(\pi, 0, \pi)$ in case 2b, respectively.

C. Case 3: $|a_f - a_0|/2 = |c_f - c_0|$

In this case, the lower bound is $n|a_f - a_0|/2 = n|c_f - c_0|$ and $c_f = c_0 \pm (a_f - a_0)/2$. Using the first two impulses given by

Eq. (52), suboptimal strategies will be constructed. The main results of this subsection are given as follows:

1) Consider the case $c_f = c_0 + (a_f - a_0)/2$ and $a_f \neq a_0$.

a) If $a_f > 0$, suboptimal two-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (\pi, \pi)$ exist.

b) If $a_f = 0$, suboptimal two-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (\pi, \pi)$ or suboptimal three-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, \pi, \pi)$ or $(0, \pi, 0)$ exist.

2) Consider the case $c_f = c_0 - (a_f - a_0)/2$ and $a_f \neq a_0$.

a) If $a_f > 0$, suboptimal two-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (0, 0)$ exist.

b) If $a_f = 0$, suboptimal two-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (0, 0)$, or suboptimal three-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (\pi, 0, \pi)$ or $(\pi, 0, 0)$ exist.

3) Consider the case $a_f = a_0$ and $c_f = c_0$.

a) Suboptimal two-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (\pi, \pi)$ or (0,0) exist.

Proof. Take two impulses given by Eq. (52) with $a_2 = a_f$. Then d and c become

$$d_2 = d_1 - 3nc_1\Delta t_2 = d_0 - 3nc_0\Delta t_1 - 3nc_1\Delta t_2,$$
$$(c_1, \bar{\alpha}_1^*) = \left(c_0 + \frac{a_0 - a_1}{2}, 0\right) \quad \text{or} \quad \left(c_0 - \frac{a_0 - a_1}{2}, \pi\right), \tag{81}$$
$$(c_2, \bar{\alpha}_2^*) = \left(c_1 + \frac{a_1 - a_f}{2}, 0\right) \quad \text{or} \quad \left(c_1 - \frac{a_1 - a_f}{2}, \pi\right)$$

First, consider the case 1a. Let $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (\pi, \pi)$. By Eq. (81), $c_2 = c_1 - (a_1 - a_f)/2 = c_0 + (a_f - a_0)/2 = c_f$. Equation (81) together with the requirement $d_2 = d_f - 3nc_f(\Delta t_1 + \Delta t_2)$ gives the equation for a_1

$$d_f = d_0 + 3n(c_f - c_0)\Delta t_1 + 3n[c_f - c_0 + (a_0 - a_1)/2]\Delta t_2 \tag{82}$$

and, hence,

$$a_1 = a_0 + 2(c_f - c_0) + (2/3n\Delta t_2)[d_0 - d_f + 3n(c_f - c_0)\Delta t_1]$$
$$= a_f + (2/3n\Delta t_2)[d_0 - d_f + 3n(c_f - c_0)\Delta t_1] = a_f + \epsilon \tag{83}$$

where, for the second equality, the assumption $c_f = c_0 + (a_f - a_0)/2$ is used, and

$$\epsilon = (2/3n\Delta t_2)[d_0 - d_f + 3n(c_f - c_0)\Delta t_1]$$

Because $a_f > 0$, $a_1 = a_f + \epsilon \geq 0$ for a large k_2. In this case, the total velocity change is

$$\Delta V_T(\Delta\dot{\mathbf{x}}^*) = (n/2)(|a_0 - a_1| + |a_1 - a_f|)$$
$$= (n/2)(|a_0 - (a_f + \epsilon)| + |\epsilon|) \leq (n/2)|a_0 - a_f| + n|\epsilon|$$

Hence, ΔV_T can be made arbitrarily close to the infimum $n|a_f - a_0|/2$.

If $a_f = 0$ and $\epsilon > 0$, then $a_1 > 0$ and the same conclusion holds. However, if $a_f = 0$, then $c_f = c_0 - a_0/2 < c_0$. Thus, the suboptimal three-impulse strategies with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*, \bar{\alpha}_3^*) = (0, \pi, \pi)$ or $(0, \pi, 0)$ in case 2, 1b can be employed. This proves 1b.

Now, consider case 2a. In this case, set $(\bar{\alpha}_1, \bar{\alpha}_2) = (0, 0)$. Again, Eq. (81) gives $c_2 = c_f$. Now Eq. (82) is replaced by

$$d_f = d_0 + 3n(c_f - c_0)\Delta t_1 + 3n[c_f - c_0 - (a_0 - a_1)/2]\Delta t_2 \tag{84}$$

and Eq. (83) by

$$a_1 = a_0 - 2(c_f - c_0) - (2/3n\Delta t_2)[d_0 - d_f + 3n(c_f - c_0)\Delta t_1]$$
$$= a_f - (2/3n\Delta t_2)[d_0 - d_f + 3n(c_f - c_0)\Delta t_1] \equiv a_f - \epsilon \tag{85}$$

A simple modification of this proof establishes case 2.

Finally, consider case 3. Suppose $a_0 = a_f > 0$. Employ the two-impulse strategy of 1a with $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (\pi, \pi)$. Then Eq. (82) becomes

$$d_f = d_0 + 3n(a_0 - a_1)/2\Delta t_2$$

and

$$a_1 = a_0 + 2(d_0 - d_f)/(3n\Delta t_2) \tag{86}$$

Hence, a_1 is positive and arbitrarily close to a_0 for a large k_2, and ΔV_T can be made arbitrarily small.

Suppose $a_0 = a_f = 0$. In this case, a_1 given by Eq. (86) is positive if $d_f < d_0$ and is arbitrarily small for large k_2. On the other hand, if $d_f > d_0$, choose $(\bar{\alpha}_1^*, \bar{\alpha}_2^*) = (0, 0)$. Then

$$d_f = d_0 - 3n(a_0 - a_1)/2\Delta t_2$$

and the same conclusion holds. Hence, in either case, suboptimal two-impulse strategies can be constructed. This completes the proof of all results 1–3.

V. Asymptotic Transfer by Feedback Control

In the previous section, the minimum-fuel problem for in-plane motion has been considered, and the optimal three-impulse controls are obtained. They are open-loop controls and depend on the exact knowledge of the system and its state. They are not able to cope with uncertainties or disturbances. Furthermore, the relative orbits of the HCW equations are not those for the nonlinear equations of the relative motion (1–3). To transfer to a relative orbit of nonlinear equations, the initialization process [25] could be used. But in this section, feedback controls, which fulfill the asymptotic relative orbit transfer and keep the final orbit, with small total velocity change are proposed for both the in-plane and out-of-plane motion. The design is based on the linearized equations.

A. Asymptotic Transfer for In-Plane Motion

For simplicity, $nt_0 = \alpha_0$ with $0 \leq \alpha_0 \leq \pi$ is assumed so that the position of the chaser at $t = 0$ is $\mathbf{x}_1(0) = [a_0 \quad 0 \quad 0 \quad -2a_0 n]^T$. Then the initial position of the chaser at t_0 is $\mathbf{x}_{10} = e^{A_1 t_0}\mathbf{x}_1(0)$, where t_0 is the initial time to start control action.

Because all of the impulse times for the optimal and suboptimal strategies in the previous section are of the form $t_j = jT/2$, the in-plane motion with impulse inputs $\Delta\dot{\mathbf{x}}_j$ at time $t_j = jT/2$ is considered:

$$\dot{\mathbf{x}}_1 = A_1\mathbf{x}_1 \quad (t \neq jT/2), \quad \Delta\mathbf{x}_1(jT/2) = B_1\Delta\dot{\mathbf{x}}_j, \quad \mathbf{x}_1(t_0) = \mathbf{x}_{10} \tag{87}$$

To design feedback controllers, a virtual vehicle is introduced in the final orbit, which is given by

$$\dot{\mathbf{x}}_2 = A_1\mathbf{x}_2, \qquad \mathbf{x}_2(t_0) = \mathbf{x}_{20}$$

where $\mathbf{x}_{20}$ is the initial condition of the virtual vehicle on the final orbit. Then the error vector $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ satisfies the equation

$$\dot{\mathbf{x}} = A_1\mathbf{x} \quad (t \neq jT/2), \quad \Delta\mathbf{x}(jT/2) = B_1\Delta\dot{\mathbf{x}}_j, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{88}$$

where $\mathbf{x}_0 = \mathbf{x}_{10} - \mathbf{x}_{20}$. Now set $\mathbf{x}_j = \mathbf{x}(jT/2^+)$, $A_{1d} = e^{A_1 T/2}$, and discretize Eq. (88) to obtain

$$\mathbf{x}_{j+1} = A_{1d}\mathbf{x}_j + B_{1d}\Delta\dot{\mathbf{x}}_j \tag{89}$$

where

$$A_{1d} = \begin{bmatrix} 7 & 0 & 0 & 4/n \\ -6\pi & 1 & -4/n & -3\pi/n \\ 0 & 0 & -1 & 0 \\ -12n & 0 & 0 & -7 \end{bmatrix}, \qquad B_{1d} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{90}$$

Recall that a linear discrete-time system (A, B) is NCVE if any initial state of the system can be steered to the origin by a control with an arbitrarily small l^2 norm [26,28]. The necessary and sufficient conditions for this are that (A, B) is controllable and $|\lambda| \le 1$ for any eigenvalue λ of A. Because (A_{1d}, B_{1d}) is controllable and the eigenvalues of A_{1d} are $(1, 1, -1, -1)$, the system equation (89) is NCVE. In fact, if (A_1, B_1) is NCVE, so is (A_{1d}, B_{1d})[28]. To exploit this property, consider the linear quadratic regulator problem given by the cost function

$$J(\Delta\dot{\mathbf{x}}, \mathbf{x}_0) = \sum_{j=1}^{\infty}[\mathbf{x}_j^T Q \mathbf{x}_j + \Delta\dot{\mathbf{x}}_j^T(j) R \Delta\dot{\mathbf{x}}_j]$$

where $Q \ge 0$ and $R > 0$ and $(\sqrt{Q}, A_{1d})$ is assumed to be observable. Then, there exists a unique positive definite stabilizing solution of the algebraic Riccati equation (ARE):

$$X = A_{1d}^T X A_{1d} + Q - A_{1d}^T X B_{1d}(R + B_{1d}^T X B_{1d})^{-1} B_{1d}^T X A_{1d}$$

The optimal control is given by the stabilizing feedback (impulse) control

$$\Delta\dot{\mathbf{x}}_j^* = K_d \mathbf{x}_j, \qquad K_d = -(R + B_{1d}^T X B_{1d})^{-1} B_{1d}^T X A_{1d} \tag{91}$$

Because the feedback law Eq. (91) is stabilizing, $\mathbf{x}_j \to 0$ as $j \to \infty$. Note that $X \to 0$ as $Q \to 0$. Moreover, the l^2 norm of the feedback control $\Delta\dot{\mathbf{x}}^*$ goes to zero. Hence, the chaser tracks the virtual vehicle asymptotically, and the asymptotic transfer to the final orbit is assured. The total velocity change of the feedback control is identified with the l^1 norm and, in general, it decreases as $Q \to 0$. Hence, by choosing a small Q, feedback controllers with good performance are obtained. For this design, a large R could be used instead of a small Q.

B. Asymptotic Transfer for Out-of-Plane Motion

Suppose the chaser lies on the orbit b_0 initially and that $nt_0 = \beta_0$. Then, at $t = 0$, $\mathbf{x}(0) = [\,b_0 \quad 0\,]^T$. Because all impulse times for optimal strategies in this case are of the form $t_j = jT/2$, a natural choice for the impulse interval for feedback control is $T/2$. However, $e^{A_2 T/2} = -I$ (identity matrix), and $(e^{A_2 T/2}, B_2)$ is not controllable. Hence, the impulse interval $T/4$ is adopted. Now

$$A_{2d} \equiv e^{A_2 T/4} = \begin{bmatrix} 0 & 1/n \\ -n & 0 \end{bmatrix}$$

and (A_{2d}, B_{2d}) is controllable. The discrete-time system (A_{2d}, B_{2d}) is NCVE, and feedback controllers can be designed by the regulator theory as in the case of the in-plane motion.

VI. Simulation Results

In our simulation, the height h_c of the circular orbit of the target spacecraft is assumed to be 500 km. Then, the period of this circular orbit is $T = 5676$ s (1.58 h) and the orbit rate is $n = 1.1068 \times 10^{-3}$ rad/s as given in Table 1, which also includes the radius of the Earth (R_e), the gravitational constant of the Earth (μ_e).

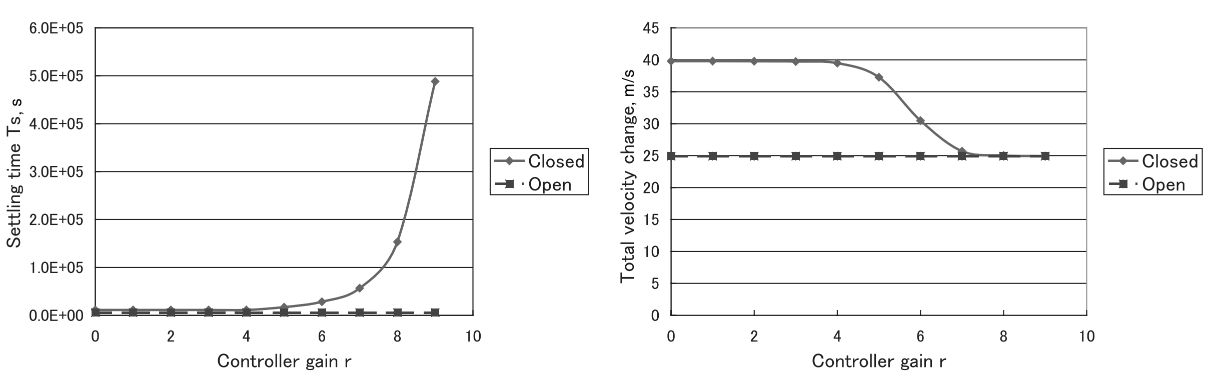
Table 1 Constants and common parameters

Constants	Values
R_e	6378.136 km
μ_e	398, 601 km^3/s^2
Common parameters	Values
h_c	500 km
n	1.1068×10^{-3} rad/s
T	5676 s (1.58 h)

A. Simulation Results for In-Plane Motion

The parameters of the initial and final orbits are assumed to be $(a_0, d_0) = (50, 50)$ (km) and $(a_f, d_f) = (5, 0)$ (km), respectively, and the initial phases of the chaser and the virtual vehicle are assumed to be $(\alpha_0, \alpha_f) = (\pi, \pi)$. These orbits approximately correspond to the elliptic orbits with an eccentricity of 0.0073 and 0.0008 in the inertial frame. Because $\alpha_0 = \pi$, it follows that $t_0 = T/2$. Hence, the chaser's initial phase is one of the optimal phases, and the first impulse time is t_0, that is, $\Delta t_1 = 0$ s. The chaser is initially at $(x_{10}, y_{10}) = (-50, 50)$, whereas the virtual vehicle is at $(x_{20}, y_{20}) = (-5, 0)$. To design a feedback impulse controller, the matrices $Q = I$ and $R = 10^r I$ in the ARE are assumed. To introduce a stopping rule for simulation, let $d_{\min}$ denote the minimum distance of a point in the final orbit from its center and $v_{\min}$ the minimum velocity of the chaser in the final orbit. The chaser is regarded in the final orbit when $|x|, |y| < 10^{-3} \times d_{\min}$ and $|\dot{x}|, |\dot{y}| < 10^{-3} \times v_{\min}$. The time required to get on the final orbit is called the settling time and is denoted by T_s. To see that Eq. (89) is NCVE, T_s and the total velocity change (l^1 norm) of the closed-loop controller are calculated for $r = 0$–9 (see Fig. 4). The total velocity change approaches the minimum total velocity change at the expense of the transfer time. The feedback controller for $r = 7(R = 10^7 I)$, which gives $T_s = 10T$ (15.8 h), is selected and applied to Eqs. (4) and (5).

First, consider the three-impulse transfer. For this example $k^* = 0.736$ and $(k_1, k_2, k_3) = (0, 1, 1)$ are admissible and correspond to the minimal time transfer. The optimal three-impulse strategy brings the chaser to the final orbit after the third impulse at $t_3 = 3T/2$. Thus, the transfer time is $T_s = T$. Its total velocity change is $\Delta V_T^* = 24.9028$ m/s, and the control inputs $(\Delta\dot{y}_1^*, \Delta\dot{y}_2^*, \Delta\dot{y}_3^*) = (-3.2899, 12.4514, -9.1616)$ (m/s^2) are found in Table 2, in which the data of the initial and final orbits are also given. The controlled trajectory associated with the optimal three-impulse controller is given in Fig. 5. Now apply the feedback controller to the in-plane motion (87). The solid line indicates the controlled trajectory of the chaser in Fig. 5 and the control inputs in Fig. 6, respectively. The settling time is $T_s = 10T = 56,760$ s (15.8 h) and the total velocity change of the closed-loop control is 25.6909 m/s. The maximum values of the inputs $\Delta\dot{x}$ and $\Delta\dot{y}$ denoted by $\Delta\dot{x}_{\max}$ and $\Delta\dot{y}_{\max}$ are



a) b)

Fig. 4 Settling time T_s and total velocity change.

Table 2 Elliptic orbits and performance indices

Parameters	Values
(a_0, d_0, α_0)	(50 km, 50 km, π rad)
(a_f, d_f, α_f)	(5 km, 0 km, π rad)
Δt_1	0 s
Performance for linearized equations	
Closed-loop controller	
ΔV_T	25.6906 m/s
T_s	56,760 s (15.8 h)
$\Delta \dot{x}_{\max}$	1.71297 m/s^2
$\Delta \dot{y}_{\max}$	8.68181 m/s^2
Open-loop controller	
ΔV_T^*	24.9028 m/s
T_s	5676 s (1.58 h)
$(\Delta \dot{y}_1^*, \Delta \dot{y}_2^*, \Delta \dot{y}_3^*)$	$(-3.2899, 12.4514, -9.1616)$ m/s^2
Performance for nonlinear equations	
ΔV_T	25.8392 m/s
T_s	56,760 s (15.8 h)
$\Delta \dot{x}_{\max}$	1.71297 m/s^2
$\Delta \dot{y}_{\max}$	9.02290 m/s^2

$(1.71297, 8.68181)$ (m/s^2). They are also important performance indices, because they are factors that determine the specifications for a thruster. All of these performance indices are given in Table 2. The impulse $\Delta \dot{x}$ in Fig. 6 remains small, and the total velocity change of the closed-loop controller is about the same as the minimum velocity change of the optimal three-impulse controller. As expected, $\Delta \dot{x}_{\max}$ and $\Delta \dot{y}_{\max}$ are smaller than $\max_j |\Delta \dot{y}_j^*|$.

Setting $z = 0$, both the optimal three-impulse controller and the feedback controller are applied to the original nonlinear equations (1) and (2). The controlled trajectory of the chaser is given in Fig. 7, and the performance indices are given in Table 2. The optimal open-loop controller does not fulfill the orbit transfer, because the final orbit is not a periodic orbit of the nonlinear equations. After the third impulse, the chaser starts drifting if the simulation is continued. On the other hand, the closed-loop controller accomplishes the orbit transfer with a little extra velocity change. Its total velocity change is 25.8392 m/s and is close to that of the HCW equations. Hence, the linear feedback controller works well for the original nonlinear systems.

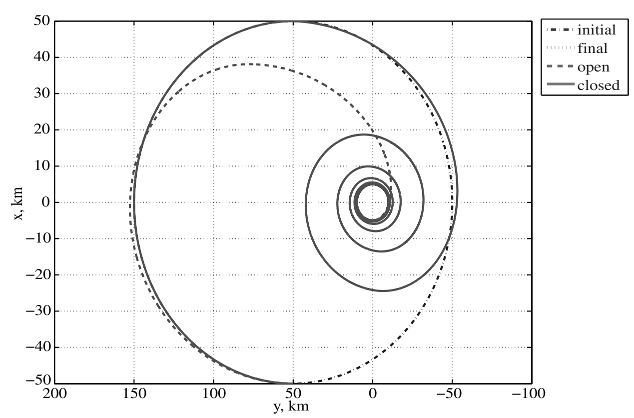


Fig. 5 Trajectories of the linear equations.

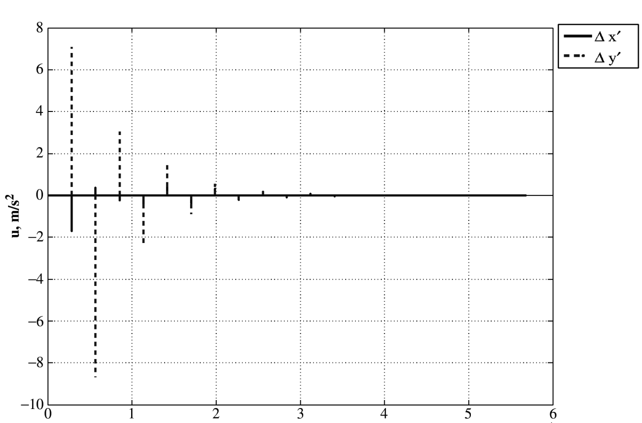


Fig. 6 Control inputs of the linear equations.

B. Simulation Results for Out-of-Plane Motion

The size parameter b of the initial and final orbits is assumed to be $(b_0, b_f) = (50, 0)$ (km) and the phase parameter is assumed to be $(\beta_0, \beta_f) = (7\pi/4, 0)$. The first impulse time is then $t_{z1}^* = 3T/8$ s. The matrices Q and R in the ARE are assumed to be $Q = I$ and $R = 1$, where I is the identity matrix of the appropriate dimension. The feedback controller is applied to the linearized equation (6) and to the nonlinear equation (3) with $x = y = 0$. As a stopping rule, the chaser is regarded to be in the final orbit when $|z| < 10^{-2}$ and $|\dot{z}| < n \times 10^{-2}$. As both responses are similar, only the trajectory of the nonlinear equation and the control inputs are depicted in Figs. 8 and 9, respectively. The transfer time is $T_s = 36194$ s (10.0 h). The velocity change of the optimal impulse is $\Delta V_T^* = 55.3397$ m/s. The total velocity change of the feedback controller is 55.9433 m/s, which is almost equal to ΔV_T^*. The maximum values of the inputs u_z denoted by $\Delta \dot{z}_{\max}$ are about 60% of ΔV_T^*. All of these performance indices are given in Table 3.

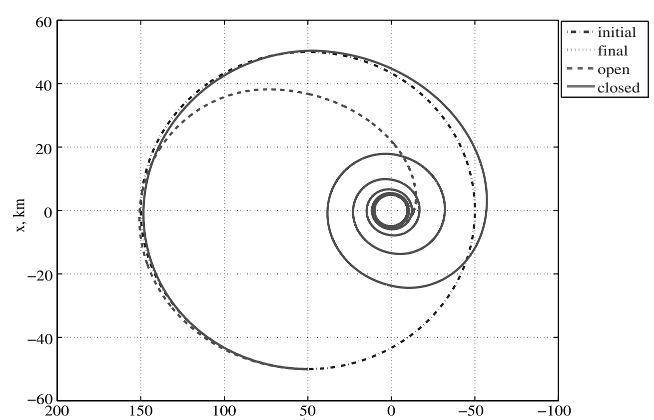


Fig. 7 Trajectory of the nonlinear equations.

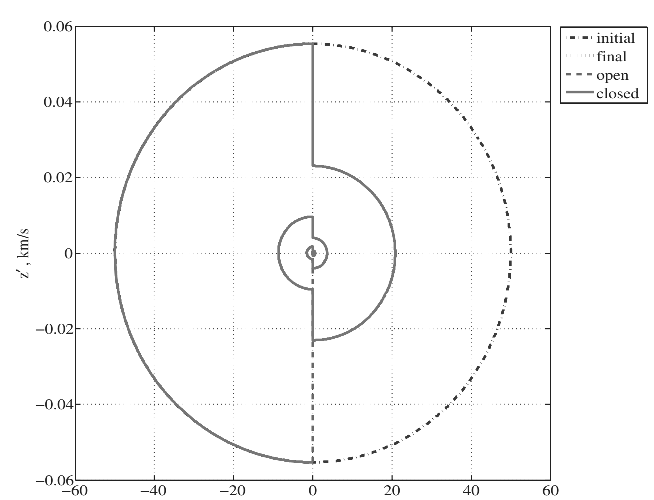


Fig. 8 Trajectory of the nonlinear equations.

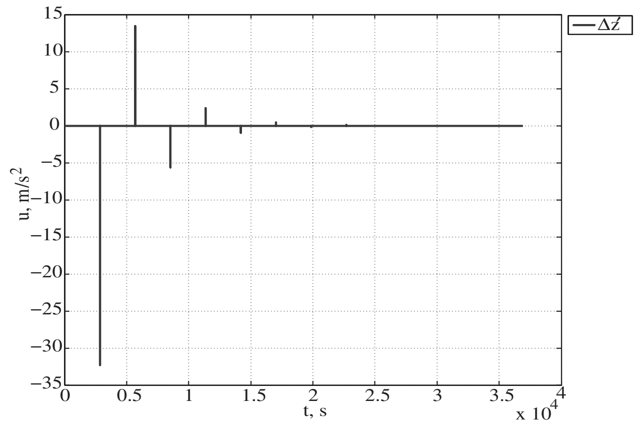


Fig. 9 Control inputs.

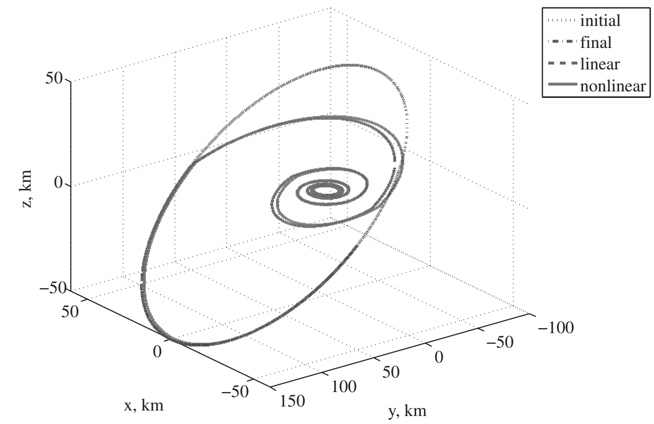


Fig. 10 Three-dimensional trajectory of the chaser.

C. Simulation Results for Full Motion

Here feedback controllers for the in-plane and out-of-plane motion are combined and applied to the nonlinear HCW Eqs. (1–3). The parameters for simulation are set as $(a_0, d_0, a_f, d_f) = (50, 50, 10, 0)$ (km), $(\alpha_0, \alpha_f) = (7\pi/4, 7\pi/4)$, $(b_0, b_f) = (50, 0)$ (km), and $(\beta_0, \beta_f) = (7\pi/4, 7\pi/4)$ (see Table 4).

Table 3 Performance indices for out-of-plane motion

Parameters	Values
(b_0, b_f)	(50 km, 0 km)
(β_0, β_f)	($7\pi/4$ rad, $7\pi/4$ rad)
t_1^*	2128 s (35.5 min)
	Performance for the linearized equations
	Closed-loop controller
ΔV_T	55.9429 m/s
T_s	36,194 s (10.1 h)
$\Delta\dot{z}_{\max}$	34.1973 m/s^2
	Open-loop controller
ΔV_T^*	55.3397 m/s
T_s	2138 s (35.6 min)
$\Delta\dot{z}_1$	55.3397 m/s^2
ΔV_T	55.9433 m/s
T_s	36,194 s (10.1 h)
	Performance for nonlinear equations
$\Delta\dot{z}_{\max}$	34.1981 m/s^2

Table 4 Performance indices for three-dimensional motion

Parameters	Values
(a_0, d_0, a_f, d_f)	(50 km, 50 km, 10 km, 0)
(α_0, α_f)	($7\pi/4$ rad, $7\pi/4$ rad)
$(b_0, \beta_0, b_f, \beta_f)$	(50 km, $7\pi/4$ rad, 0 km, $7\pi/4$ rad)
t_1^*	710 s (11.8 min)
t_{z1}^*	2128 s (35.5 min)
	Performance for the linearized equations
ΔV_T	79.2660 m/s
T_s	45,024 s (12.5 h)
$\Delta\dot{x}_{\max}$	0.112185 m/s^2
$\Delta\dot{y}_{\max}$	10.0327 m/s^2
$\Delta\dot{z}_{\max}$	34.2018 m/s^2
	Performance for the nonlinear equations
ΔV_T	92.0445 m/s
T_s	45,025 s (12.5 h)
$\Delta\dot{x}_{\max}$	9.01859 m/s^2
$\Delta\dot{y}_{\max}$	5.77733 m/s^2
$\Delta\dot{z}_{\max}$	33.5825 m/s^2
	Minimum total velocity change
ΔV_T^*	77.4756 m/s^2

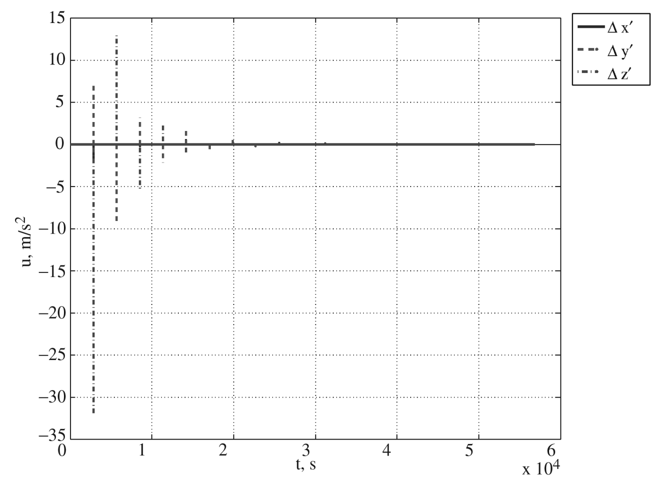


Fig. 11 Control inputs.

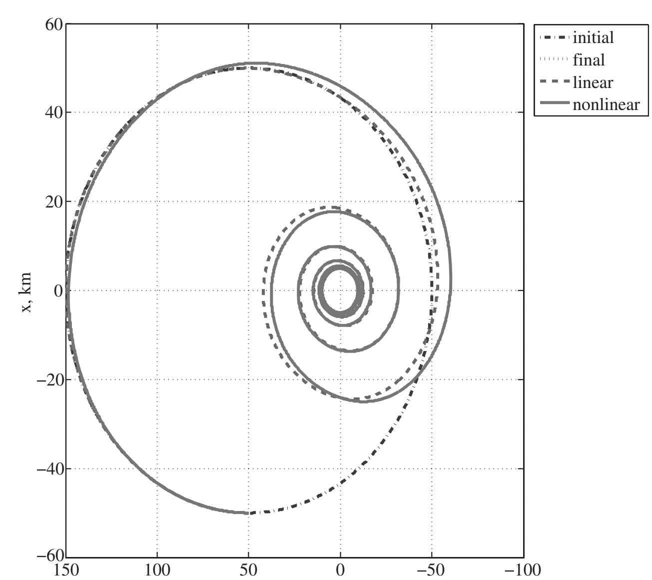


Fig. 12 In-plane motion.

The first impulse time is $t_1^* = T/8 = 11.8$ min for the in-plane motion and $t_{z1}^* = 3T/8 = 35.5$ min for the out-of-plane motion, respectively. The feedback controller is applied to the linearized equations (4–6) as well as to the nonlinear equations (1–3). The total velocity change of the feedback controller is 79.2660 m/s for the

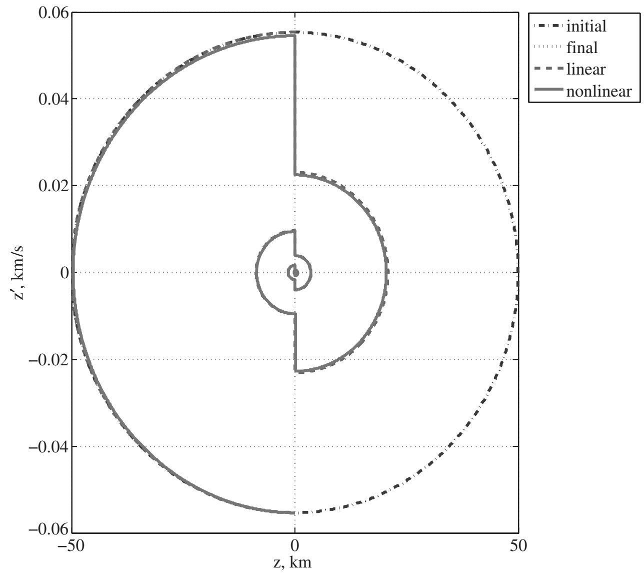


Fig. 13 Out-of-plane motion.

linearized equations, whereas it becomes 92.0445 m/s for the nonlinear equations. Other performance indices are given in Table 4. The controlled trajectory of the nonlinear equations in the (x, y, z) space are depicted in Fig. 10, and the control inputs are depicted in Fig. 11. The trajectory of the HCW equations is also included. The projections of the trajectories onto the (x, y) plane and the $(z, \dot{z})$ plane are given in Figs. 12 and 13, respectively.

VII. Conclusions

In this paper, the relative orbit transfer problem associated with the HCW equations is considered. The minimum-fuel problem with impulsive controls is formulated as an open-time problem. For the in-plane motion, two lower bounds for the minimum total velocity change are established, and optimal impulse strategies that attain the lower bound are constructed when the difference of the size parameters is large compared with that of drift parameters. It is shown that the impulse times for the optimal strategies are those instants at which the velocity in the radial direction is zero. If the difference of the drift parameters is larger than that of the size parameters, only suboptimal strategies with two or three impulses are assured, whose total velocity changes are arbitrarily close to the lower bound.

For the out-of-plane motion, optimal single impulses exist, and their impulse times are those instants at which the chaser crosses the orbit plane of the target.

Based on these results and the null controllability with vanishing energy of the Hill–Clohessy–Wiltshire equations, feedback controllers with a total velocity change close to the optimal one are proposed. Numerical simulations for the three-dimensional relative motion are presented, and a feedback controller with good performance is obtained.

Acknowledgments

The research of the second author is partly supported by the Ministry of Education, Sports, Science and Technology, Japan under a Grant-in-Aid for Scientific Research (C), no. 19560785. The authors would like to thank the associate editor and reviewers for their valuable comments on and suggestions for the paper.

References

[1] Prussing, J. A., and Conway, B. A., *Orbital Mechanics*, Oxford Univ. Press, New York, 1993, pp. 139–152.

[2] Vallado, D. A., *Fundamentals of Astrodynamics and Applications*, 2nd ed., Microcosm, Inc., El Segundo, CA, 2001, pp. 374–399.

[3] Wie, B., *Space Vehicle Dynamics and Control*, AIAA, Reston, VA, 1998, pp. 282–285.

[4] Prussing, J. E., "Optimal Four-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal*, Vol. 7, No. 5, 1969, pp. 928–935.

[5] Prussing, J. E., "Optimal Two- and Three-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal*, Vol. 8, No. 7, 1970, pp. 1221–1228.

[6] Jezewsky, D. J., and Donaldson, J. D., "An Analytic Approach to Optimal Rendezvous Using Clohessy–Wiltshire Equations," *Journal of the Astronautical Sciences*, Vol. 27, No. 3, 1979, pp. 293–310.

[7] Carter, T. E., "Optimal Impulsive Space Trajectories Based on Linear Equations," *Journal of Optimization Theory and Applications*, Vol. 70, No. 2, 1991, pp. 277–297.
doi:10.1007/BF00940627

[8] Carter, T. E., and Alvarez, S. A., "Quadratic-Based Computation of Four-Impulse Optimal Rendezvous near Circular Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 1, 2000, pp. 109–117.

[9] Vassar, R. H., and Sherwood, R. B., "Formationkeeping for a Pair of Satellites in a Circular Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 2, 1985, pp. 235–242.

[10] Redding, D. C., Adams, N., and Kubiak, E. T., "Linear-Quadratic Stationkeeping for the STS Orbiter," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 2, 1989, pp. 248–255.

[11] Kapila, V., Sparks, A. G., Buffington, J. M., and Yan, Q., "Spacecraft Formation Flying: Dynamics and Control," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 3, 2000, pp. 561–564.

[12] Kang, W., Sparks, A., and Banda, S., "Coordinate Control of Multisatellite Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 2, 2001, pp. 360–368.

[13] Campbell, M. E., "Planning Algorithm for Multiple Satellite Clusters," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 5, 2003, pp. 770–780.

[14] Vaddi, S. S., Alfriend, K. T., Vadali, S. R., and Sengupta, P., "Formation Establishment and Reconfiguration Using Impulsive Control," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 2, 2005, pp. 262–268.
doi:10.2514/1.6687

[15] Carter, T. E., and Humi, M., "Fuel-Optimal Rendezvous near a Point in General Keplerian Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 567–573.

[16] Carter, T. E., and Brient, J., "Fuel-Optimal Rendezvous for Linearized Equations of Motion," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 6, 1992, pp. 1411–1416.

[17] Carter, T. E., and Brient, J., "Linearized Impulsive Rendezvous Problem," *Journal of Optimization Theory and Applications*, Vol. 86, No. 3, Sept. 1995, pp. 553–584.
doi:10.1007/BF02192159

[18] Carter, T. E., "State Transition Matrices for Terminal Rendezvous Studies: Brief Survey and New Example," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 148–155.

[19] Wiesel, W. E., "Optimal Impulsive Control of Relative Satellite Motion," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 1, 2003, pp. 74–78.

[20] Neustadt, L. W., "Optimization, a Moment Problem, and Nonlinear Programming," *SIAM Journal on Control*, Vol. 2, No. 1, 1964, pp. 33–53.

[21] Neustadt, L. W., "A General Theory of Minimum-Fuel Space Trajectories," *SIAM Journal on Control*, Vol. 3, No. 2, 1965, pp. 317–356.

[22] Prussing, J. E., "Optimal Impulsive Linear Systems: Sufficient Conditions and Maximum Number of Impulses," *Journal of the Astronautical Sciences*, Vol. 43, No. 2, 1995, pp. 195–206.

[23] Inalhan, G., Tillerson, M., and How, J. P., "Relative Dynamics and Control of Spacecraft Formations in Eccentric Orbits," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 1, 2002, pp. 48–59.

[24] Ulybyshev, Y., "Long-Term Formation Keeping of Satellite Constellation Using Linear-Quadratic Controller," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, 1998, pp. 109–115.

[25] Gurfil, P., "Relative Motion Between Elliptic Orbits: Generalized Boundedness Conditions and Optimal Formationkeeping," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 4, 2005, pp. 761–767.
doi:10.2514/1.9439

[26] Shibata, M., and Ichikawa. A., "Orbital Rendezvous and Flyaround Based on Null Controllability with Vanishing Energy," *Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 4, July–Aug. 2007, pp. 934–945.
doi:10.2514/1.24171

[27] Priola, E., and Zabczyk, J., "Null Controllability with Vanishing Energy," *SIAM Journal on Control and Optimization*, Vol. 42, No. 3, 2003, pp. 1013–1032.

doi:10.1137/S0363012902409970
[28] Ichikawa, A., "Null Controllability with Vanishing Energy for Discrete-Time Systems," *Systems and Control Letters*, Vol. 57, No. 1, 2008, pp. 34–38.
doi:10.1016/j.sysconle.2007.06.008
[29] Ichikawa, A., "Null Controllability with Vanishing Energy for Discrete-Time Systems in Hilbert Space," *SIAM Journal on Control and Optimization*, Vol. 46, No. 2, 2007, pp. 683–693.
doi:10.1137/060657637
[30] Palmer, P., "Optimal Relocation of Satellites Flying in Near-Circular-Orbit Formations," *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 3, 2006, pp. 519–526.
doi:10.2514/1.14310